On Nearly Umbilical Hypersurfaces

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch–naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Daniel Raoul Perez

aus

Österreich

Promotionskomitee

Prof. Dr. Camillo De Lellis (Leiter der Dissertation)
Prof. Dr. Thomas Kappeler

Zürich, 2011

Diese Arbeit ist all jenen gewidmet, die nie in den Genuss einer höheren Ausbildung kamen, obwohl sie weitaus mehr Talent gehabt hätten als ich.

A university is very much like a coral reef. It provides calm waters and food particles for delicate yet marvellously constructed organisms that could not possibly survive in the pounding surf of reality, where people ask questions like 'Is what you do of any use?' and other nonsense.

Terry Pratchett, Ian Stewart and Jack Cohen,
The Science of Discworld, revised ed.,
Ebury Press, London, 2002, 150–151

Zusammenfassung

Im Jahr 2005 erzielten C. De Lellis und S. Müller eine quantitative Stabilitätsaussage über den klassischen Nabelpunktsatz der Differentialgeometrie. Für glatte, geschlossene und zusammenhängende Flächen Σ im $\mathbb{R}^3$, bewiesen sie folgende Abschätzung im kritischen Exponenten Zwei:

([**DLM05**, (1)]) $$\inf_{\lambda \in \mathbb{R}} \|A - \lambda \operatorname{id}\|_{L^2(\Sigma)} \leq C \left\| A - \tfrac{\operatorname{tr} A}{2} \operatorname{id} \right\|_{L^2(\Sigma)} .$$

Dabei bezeichnet A die zweite Fundamentalform von Σ, und $C > 0$ ist eine von der Fläche unabhängige Konstante.

Ziel der vorliegenden Arbeit ist es, obige Abschätzung auf höhere Dimensionen n, sowie allgemeine, nicht–kritische Exponenten p der Lebesgue–Norm zu erweitern. Wir betrachten glatte, geschlossene und zusammenhängende Hyperflächen des $\mathbb{R}^{n+1}$ und unterscheiden die beiden Fälle $1 < p \leq n$ und $p > n$. Im ersten Fall gelingt uns die Verallgemeinerung für konvexe Hyperflächen, wohingegen wir im zweiten Fall die zusätzliche Annahme treffen müssen, dass die L^p–Norm der zweiten Fundamentalform einer (von uns bestimmbaren) Schranke genügt.

Des Weiteren ermitteln wir in obiger L^2–Ungleichung die optimale Konstante für konvexe Hyperflächen des $\mathbb{R}^{n+1}$.

Schliesslich beweisen wir noch die Notwendigkeit der Konvexitätsannahme, wann immer $1 \leq p < n$ ist.

Abstract

In 2005, C. De Lellis and S. Müller obtained a quantitative rigidity result regarding the classical theorem of differential geometry about surfaces all whose points are umbilical. For smooth, closed and connected surfaces Σ in $\mathbb{R}^3$, they proved the following estimate in the critical exponent two:

$$\inf_{\lambda\in\mathbb{R}} \|A - \lambda\,\mathrm{id}\|_{L^2(\Sigma)} \leq C \left\| A - \tfrac{\operatorname{tr} A}{2}\,\mathrm{id} \right\|_{L^2(\Sigma)} . \qquad \text{([**DLM05**, (1)])}$$

Here, A denotes the second fundamental form of Σ, and $C > 0$ is a constant which is independent of the surface.

The goal of the present work is to generalise the above estimate to higher dimensions n and general non–critical exponents p of the Lebesgue norm. We consider smooth, closed and connected hypersurfaces in $\mathbb{R}^{n+1}$ and distinguish the two cases $1 < p \leq n$ and $p > n$. In the first case, we obtain the generalisation for convex hypersurfaces, whereas in the second we need to make the additional assumption that the L^p–norm of the second fundamental form satisfy some bound (which we are able to preset).

Furthermore, we establish the optimal constant in the above L^2–inequality for convex hypersurfaces in $\mathbb{R}^{n+1}$.

Finally, we also prove that the hypothesis of convexity is necessary, whenever $1 \leq p < n$.

Contents

Preface

Contents

1. Introduction

1.1. The Nabelpunktsatz. A point on a smooth surface in Euclidean space is called *umbilical* if the two principal curvatures in this point coincide. According to the classical *Nabelpunktsatz* (German for "umbilical theorem"), if all the points of a smooth, connected surface are umbilical, then this surface is either a subset of a plane or of a sphere (see, e.g., [**dC76**, §3.2, Prop.4, p.147], [**Str88**, §3.5(2), p.122], [**Küh08**, Thm.3.14, p.51] or [**Pre10**, Prop.8.2.9, p.191]). This is one of the first rigidity results in differential geometry of type *local to global*, in that a local (or even pointwise) condition has a global consequence. The word *rigidity* can be interpreted in two ways here. Either by saying that the two obvious cases are the only ones occurring, or by pointing out that the curvature cannot vary across the surface.

There are many generalisations of the *Nabelpunktsatz* in various directions. For instance, there is an n–dimensional version (see [**Spi99**, Lem.1, p.8]), and the smoothness hypothesis can be weakened (see, e.g., [**BF36**, §5], [**Har47**, Thm.1] and [**Pau08**]). Also, there are versions applying to submanifolds of higher co–dimension in $\mathbb{R}^{n+1}$ (see [**Spi99**, Thm.26, p.75]), or even in spaces of constant curvature (see [**Spi99**, Thms.27&29, pp.75&77]).

A natural question to ask about all rigidity results is whether the conclusion is *stable*. For the *Nabelpunktsatz* this shall mean: does the assumption that all points of a closed surface be *nearly* umbilical entail that this surface must be *nearly* a sphere? Here we exclude the planar conclusion by restricting attention to *closed* surfaces, i.e. compact ones without boundary. Of course, one needs to give a more precise meaning to the word *nearly*, both in the hypotheses and in the conclusion. To do this, we observe that the ratio of the principal curvatures of a sphere is identically one. Thus, for a strictly convex surface, we obtain a rough measure for a point q not to be umbilical by considering $\eta(q) = \frac{\lambda_{\max}(q)}{\lambda_{\min}(q)} - 1$, where $\lambda_{\min}(q)$ and $\lambda_{\max}(q)$ denote the minimal and maximal principal curvature in q, respectively. Note that we assume strict convexity in order to ensure that $\eta(q)$ is well–defined everywhere. On the other hand, we can say that the surface is close to a sphere, if it lies in a thin spherical shell, and we measure that by considering the difference $\rho = \frac{R}{r} - 1$, where $0 < r < R$ are the radii of the two bounding spheres. Now we can ask:

- Does uniform smallness of η imply smallness of ρ? (*qualitative question*)
- Is there a universal constant $C > 0$ such that $\rho \leq C \sup_{q\in\Sigma} \eta(q)$, for all compact, strictly convex surfaces $\Sigma \subset \mathbb{R}^3$? (*quantitative question*)

These questions are obviously well–posed in higher dimensions, as well.

1.2. The Russian school. A. V. Pogorelov in [**Pog67**], complemented by H. Guggenheimer [**Gug69**], answers both of these questions in the affirmative — the monograph [**Pog73**, §VII.9] revisits these results. Here, one should also mention the work by Yu. E. Borovskiĭ [**Bor67**,**Bor68**], who achieves a positive answer to the first question with different methods, as well as Yu. A. Volkov and N. S. Nevmeržickiĭ ([**Vol63**], [**Nev69**] and reference 134 in [**Res94**]), whose papers we were not able to see (like A. I. Fet's related stability result [**Fet63**] — see also work by V. I. Diskant, such as [**Dis71**], as well as further references given in [**Sch89**]).

After these first results, several mathematicians endeavoured in generalising A. V. Pogorelov's theorem, among which D. Koutroufiotis [**Kou71**], J. D. Moore [**Moo73**] and, much later, K. Leichtweiss [**Lei99**], who all consider curvature–quantities other than the ratio between the principal curvatures (the latter author gives an explicit constant estimating the global deviation given the local one — see also R. Schneider [**Sch88**, Thm.2] and B. Andrews [**And94**, Thm.5.1&Lem.5.4], who obtain related results as corollaries).

Others took more interest in weakening the sense in which the quantity η is small. One possibility of doing so is to assume this condition to hold only in some form of average, for instance by replacing the sup–norm with an L^p–norm. Then one asks whether smallness of $\|\eta\|_{L^p}$ implies smallness of ρ, or even seeks a precise estimate of the form $\rho \leq C \|\eta\|_{L^p}$. For convex hypersurfaces of $\mathbb{R}^{n+1}$, Yu. G. Reshetnyak [**Res68**] and S. K. Vodop′yanov [**Vod70**] achieve this for a slightly different control–quantity

— a detailed exposition is given in Reshetnyak's book [**Res94**, Ch.6]. Not only do they conclude the qualitative stability of the *Nabelpunktsatz* in this weak setting, but they also obtain a quantitative estimate under a suitable smallness assumption on the right–hand side.

1.3. G. Huisken's question. In 2003, G. Huisken asked C. De Lellis and S. Müller whether one could establish a similar result when replacing $\|\eta\|_{L^2}$ by the L^2–norm of the traceless part of the second fundamental form of a smooth, closed and connected, but otherwise arbitrary surface in $\mathbb{R}^3$. The question was motivated by applications to foliations of asymptotically flat three–manifolds by surfaces of prescribed mean curvature (see [**Met07**] and subsequent works [**LMS09**, **LM10**], where the result mentioned below is crucial for ensuring that the leaves be close to spheres).

We now expose why one might want to look at this quantity. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth hypersurface endowed with the Riemannian metric g induced by the ambient Euclidean space. Recall that the *second fundamental form* A is the quadratic form on the tangent bundle whose eigenvalues are the principal curvatures, and the *mean curvature* H is its trace. Consequently, the *traceless part* $\mathring{A} = A - \frac{H}{n}g$ *of the second fundamental form* constitutes a measure for the deviation of each single principal curvature from the arithmetic mean of all of them. Clearly, an umbilical point q is then characterised by the vanishing of $\mathring{A}(q)$, and spheres are characterised by being closed and having $\mathring{A}$ vanish globally, as a consequence of the *Nabelpunktsatz*.

For a sphere of radius R, all the principal curvatures are equal to $\frac{1}{R}$, and thus, putting $\lambda = \frac{\overline{H}}{n}$, where $\overline{H}$ is the mean of H over Σ, the quadratic form $A - \lambda g$ vanishes precisely on spheres. From the viewpoint of stability, then, it is some norm of this quadratic form which one seeks to control with some norm of the traceless part of the second fundamental form. In [**DLM05**], C. De Lellis and S. Müller were able to provide such an estimate in an L^2–sense for surfaces, thereby answering G. Huisken's question in the affirmative. In addition, and under an appropriate smallness assumption on $\|\mathring{A}\|_{L^2}$, they obtained in a quantitative way the closeness to a sphere in the norm of the Sobolev space $W^{2,2}$. Only one year later, they sharpened that previous result by proving even quantitative C^0–closeness (see [**DLM06**]). Their remarkable theorems are quite involved and require a very subtle analysis. This is due to the fact that the quantities considered are measured in the so–called *critical* norm, which we now explain.

1.4. Critical, sub–critical and super–critical exponents. The way we understand the distinction between critical, sub–critical and super–critical in this context is as follows. If the L^p–norm of a quantity tends to zero when we enlarge Σ homothetically (we say we are "blowing up"), then we call the exponent p *super–critical.* If, however, that norm explodes under this rescaling, the exponent is called *sub–critical.* The *critical* case is characterised by the scaling invariance of the L^p–norm of that quantity. Intuitively, in the super–critical case, high–frequency oscillations in the quantity get over–compensated by the rescaled norm under blowing up. Put differently, our quantity might have slightly better regularity than we could originally expect. On the other hand, in the sub–critical case, the regularity might be poor. Thus, it may require additional assumptions to be able to prove theorems in the sub–critical situation which are valid in the super–critical one. The borderline case (i.e. the critical one) is, usually, the hardest one to treat, since neither of the non–critical cases gives any indication on what the minimal assumptions could be. From this point of view alone, the work of C. De Lellis and S. Müller is, indeed, astounding.

1.5. The main estimate. The aim of this thesis is to provide a generalisation of the estimate outlined above ([**DLM05**, (1)]) to arbitrary dimensions and all non–critical exponents. The sought–after result would be of the form: For every $n \geq 2$ and $p \in (1, +\infty)$, there exists a constant $C > 0$, such that, for every closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, the inequality

$$\inf_{\lambda \in \mathbb{R}} \|A - \lambda g\|_{L^p(\Sigma)} \leq C \, \|\mathring{A}\|_{L^p(\Sigma)} \tag{MAIN ESTIMATE}$$

holds. Note that this estimate is, in general, not true, but we can show it under certain additional hypotheses on Σ. A detailed description of our results is given in the next section.

1.6. Historical note. The *Nabelpunktsatz* is sometimes attributed to G. Darboux (e.g. in [**Res94**]). However, it appears that the first author to prove this result is J.–B.–C.–M. Meusnier de Laplace in his sole mathematical Mémoire [**MdL85**, Prob.III, §34, pp.500–502], presented in 1776 and printed in 1785. Moreover, Darboux himself gives credit to Meusnier regarding it ([**Dar96**, Vol.1, Ch.III.1, §175, p.270]). Meusnier, in turn, emphasises the influence of earlier work by L. Euler [**Eul67**]. G. Monge, one of Meusnier's teachers, also obtains the *Nabelpunktsatz* using related methods and at about the same time ([**Mon00**, no.26]; see also his later treatise [**Mon50**]) — this is why some authors like J. V. Boussinesq [**Bou90**, no.I.II.194, p.264] attribute the result to Monge. For further reading, we suggest [**Hil20**], [**Eis20**], [**Str33**] and [**Tru96**], as well as Darboux's Éloge on Meusnier [**Dar12**].

2. Presentation of our results

In this work, unless otherwise stated, $n \geq 2$ and $\Sigma \subset \mathbb{R}^{n+1}$ denotes a smooth, closed and connected hypersurface of $\mathbb{R}^{n+1}$. Thus, Σ is orientable. In order to simplify the presentation of our arguments, we additionally require the hypersurface to have unit n–dimensional volume. This bears no restriction, however, since all our results are easily reformulated to apply to hypersurfaces of arbitrary area.

2.1. Chapter 1. The super–critical case $p > n$. We set out to prove the main estimate for general dimensions n and exponents $p > n$, and show C^0–closeness to a sphere if $\|\mathring{A}\|_{L^p(\Sigma)}$ is small (N.B.: in contrast to the results of C. De Lellis and S. Müller, no *quantitative* estimate was obtained). Unfortunately, we could not establish the main estimate for a constant on the right–hand side that is fully independent of the hypersurface under consideration. More precisely, we prove that the estimate holds true with a constant C depending only on n, p and some $c_0 > 0$, whenever $\|A\|_{L^p(\Sigma)} \leq c_0$.

The idea of our proof is as follows. We first show that an inequality analogous to the main estimate holds locally, i.e. in "appropriate" charts. This is done by establishing a partial differential equation satisfied by the components of the second fundamental form A in terms of the components of its traceless part $\mathring{A}$. Then we invoke the classical Calderón–Zygmund inequality ([**CZ56**]), which basically tells us that the L^p–norm of the Hessian of a function can be bounded by the L^p–norm of its Laplacian. Incidentally, the use of this famous result is also a key step in S. K. Vodop′yanov's approach ([**Res94**, ch.6]), which otherwise relies on methods completely different from ours. Using the bound on $\|A\|_{L^p(\Sigma)}$, we show that we can cover Σ with a controlled number of such "appropriate" charts with large enough overlap. The global main estimate then follows from the local one.

2.2. Chapter 2. The sub–critical case $1 \leq p < n$. The idea here is to apply the same strategy as in the super–critical case. While in Chapter 1 we could conclude as described above, we need to make additional assumptions in the case at hand. It turns out that assuming convexity of Σ is sufficient.

Observing that the main estimate in the present situation is trivially fulfilled whenever its right–hand side is not small, we can assume without loss of generality that some preset bound $c_0' > 0$ on $\|\mathring{A}\|_{L^p(\Sigma)}$ be given. But in the non–super–critical, convex case this implies a bound c_0 on $\|A\|_{L^p(\Sigma)}$, thus eliminating the disappointing restriction encountered before.

However, the passage from local to global, namely the generation of a suitable covering, proved to be fairly laborious. This is due to the fact that we could no longer use the additional regularity obtained for $p > n$ to show that the desired charts have a certain minimal size. To achieve the same here, we need to rule out the possibility that the hypersurface be close to degenerate. We do this by proving that Σ must be contained in a spherical shell whose radii depend only on n, p and our preset bound c_0. Notice, though, that we did not show a quantitative version of this circumstance, i.e., we do not know how the deviation of the ratio of these two radii from one is controlled by the L^p–norm of $\mathring{A}$. Moreover, even if we feel that the non–degeneracy should hold in the case $p = 1$ as well, we could not establish it there. In turn, our considerations work equally well for proving the main estimate in the critical case $p = n$, even though requiring convexity in that situation seems exaggerated.

2.3. Chapter 3. L^2–theory. Here we treat a special non–super–critical case, namely the one when $p = 2$. Our central result is the main estimate with an explicit constant for hypersurfaces Σ of non–negative Ricci curvature (which, in the Euclidean case, is equivalent to convexity — see Proposition 3.2). The proof of this theorem is very short and based on an elegant strategy employed by C. De Lellis and P. M. Topping in [**DLT10**]. The key idea is to solve a suitable Poisson problem on Σ.

Afterwards, we exhibit a geometric flow approach due to G. Huisken (privately communicated to C. De Lellis) which is tailored to the critical two–dimensional case. This yields an alternative proof of our estimate and applies to strongly mean convex boundaries of star–shaped domains. These assumptions are, in fact, strictly weaker than ours, and they are enough to retrieve the same constant we obtained before.

We then endeavour to adapt this technique to the higher dimensional situation, where we face a sub–critical problem. Despite assuming convexity again, our reasoning only delivers a much larger constant than expected in the main estimate. Nevertheless, we find the argument instructive.

2.4. Chapter 4. Optimality. In this chapter, we subject our results to some optimality considerations. We start by demonstrating that the constant found in our L^2–estimate of Chapter 3 is optimal among Ricci–positive hypersurfaces. We do this by showing the existence of a suitable deformation of a round sphere. Afterwards, by constructing an explicit counter–example, we prove that the assumption of having non–negative Ricci curvature is optimal for *all* sub–critical exponents $p \in [1, n)$. Finally, restricting ourselves to the critical two–dimensional case, we show that our optimal constant cannot work for generic surfaces, thereby relating our result to the one of C. De Lellis and S. Müller. That last counter–example is due in part to P. M. Topping and C. De Lellis, whereas for the previous, the latter and S. Müller should be credited as well.

2.5. Complements. We conclude this work with two appendices. The first one contains three little lemmas which are all true in a slightly more general context than the one in which they are applied in the text and might be of independent interest. The second appendix, in contrast, contains work which has no effect on the topics just described. It is concerned with a preliminary step towards generalising the present results to hypersurfaces in Riemannian manifolds of non–negative Ricci curvature. More precisely, we give several L^2–integral quantities on a spherical cap or on its boundary, and calculate their second variation under a volume–preserving deformation. The chosen quantities appear both in the "Almost–Schur Lemma" of C. De Lellis and P. M. Topping ([**DLT10**]), as well as in this thesis. Admittedly, the obtained formulæ are rather unmanageable, even in a special case in which a lot of simplifications occur. In fact, not even that situation gave us a hint on how to proceed, and no result is obtained so far. We produce these calculations in spite of that, just in case someone might find them useful.

3. Discussion of our work, open problems

3.1. Weaknesses. In the previous section, we already mentioned two major shortcomings of our results, most prominently the necessity in the super–critical case to require a bound on the L^p–norm of the second fundamental form. At this point, we have no hope to overcome this restriction using our techniques. Also, we pointed out our inability to prove a C^0–estimate for the closeness to a sphere, something which we expect to be of importance in possible applications.

For this last objection, however, we already made a first step into the right direction, by concluding qualitative C^0–closeness from our main estimate. In contrast to the first drawback, then, this can be considered work in progress.

3.2. Implications. Perhaps the most satisfactory parts of the present work are the two theorems treating the sub–critical cases. Our optimality results of Chapter 4 indicate their strengths. Indeed, not only were we able to provide the best possible constant in the L^2–estimate of Chapter 3, but we also proved that assuming convexity is necessary for general sub–critical exponents, as treated in Chapter 2. Both our main theorems of these chapters are thus given under optimal hypotheses when $p < n$, and the L^2–theorem even gives an optimal statement.

In contrast, the main theorem of Chapter 1 about the general super–critical case lacks this feature. Nevertheless, it appears to step out from the usual (convex) context which we found in the literature. It would therefore seem a worthwhile goal to strengthen this result.

3.3. On the optimal constant. Also, one should be able to extract useful information by determining in general the optimal constant in the main estimate. Regarding this constant, another route one might want to pursue is a further analysis of the critical two–dimensional case. In view of the results we present, one might wonder about the most general hypotheses under which the constant we and G. Huisken found would apply. We know now that mean convex and star–shaped is sufficient, but this does not preclude the possibility of a more general condition under which the optimal constant is valid. On a related note, it would be interesting to investigate the size of the universal constant appearing in the original work of C. De Lellis and S. Müller.

3.4. Beyond Euclidean? Finally, a widely open problem is whether it is possible to combine our work with the one of C. De Lellis and P. M. Topping [**DLT10**], who prove an L^2–estimate analogous to ours (note that the inequality below appears to have been obtained already by B. Andrews in unpublished work — see [**CLN06**, §B.3, pp.517–519] for an exposition; however, [**DLT10**] also show the optimality of the constant appearing on the right–hand side). More precisely, for closed Riemannian manifolds of non–negative Ricci curvature and dimension larger than two, they prove

$$\left\|\mathrm{Ric} - \frac{\overline{\mathrm{Scal}}}{n} g\right\|_{L^2} \leq \frac{n}{n-2} \left\|\mathrm{Ric} - \frac{\mathrm{Scal}}{n} g\right\|_{L^2},$$

and show that the constant on the right–hand side is optimal (we wish to repeat at this point, that the proof of our L^2–estimate is, in fact, a simple adaptation of theirs). A first, albeit very small, step in this direction is attached as the second appendix.

Notations and conventions

The present work assumes a certain familiarity with Riemannian geometry and generally follows standard notation — for background references, the reader may consult the books recommended below.

Contents

General remarks. In the whole text, the dimension n is assumed to be at least two. The symbol Σ denotes a smooth, closed (i.e. compact, without boundary) and connected hypersurface in $\mathbb{R}^{n+1}$. The Riemannian metric g and the second fundamental form A on Σ are the ones inherited from Euclidean $\mathbb{R}^{n+1}$. We will, in general, abuse notation for objects of the tangent bundle, insofar as we will identify them with their respective push–forwards via the embedding of Σ into $\mathbb{R}^{n+1}$. Also, whenever we take norms or traces of quantities in the tensor bundle, we imply the appropriate usage of the metric g (however, we often write tr_g for the trace–operator). The same holds true for derivatives, unless stated otherwise. Moreover, whenever we put indices on quantities of the tensor bundle without specifying the chart we use, they will refer to some generic system of coordinates, and the metric g will be used to raise and lower those indices (except for some cases we will point out).

Background reading. The author got his geometric education from several different sources, among which [**BG92**], [**dC76**], [**Mil97**], [**Lee97**], [**GHL04**], [**dC92**] and [**Nic07b**]. He also liked some expositions in [**Bes87**] and [**O'N83**], and enjoyed reading parts of [**Ber03**]. For analytical questions of a general nature, he learnt a lot from [**Rud87**], [**Bar95**], [**Eva98**], [**GT01**] and [**EG92**], whereas he recommends the outstanding [**Aub98**], as well as [**Jos08**], for more specific questions on analysis on manifolds (but see also the introductory text [**Spi65**]). Regarding convexity, he found [**Sch93**] extremely helpful, as well as [**Roc70**]. Finally, for topics in functional analysis, he usually uses [**Bre83**] and [**Rud91**].

List of symbols. What follows is a list of generic symbols which we will use frequently and, on many occasions, without specifying their meaning again. We sometimes use subscripts to emphasise the context to which these symbols belong.

$B_r(x)$	The ball of radius r around x in the ambient space (usually $\mathbb{R}^{n+1}$)
$D_r(y)$	The ball of radius r around y in coordinate space (normally $\mathbb{R}^n$)
vol_m	The m–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$
$\langle\,\cdot\,,\,\cdot\,\rangle_P$	The Euclidean scalar product in a linear subspace $P \subset \mathbb{R}^{n+1}$
id	The identity $(1,1)$–tensor
$\delta^i{}_j$	The Kronecker delta
D	The coordinate derivative of Euclidean space
g	The Riemannian metric of Σ
$d\mathrm{vol}_g$	The volume form associated to g
$\lvert\cdot\rvert_g$	The Hilbert–Schmidt norm with respect to g, acting on sections of the tensor bundle — we usually drop the subscript g
tr_g	The trace with respect to g
∇	The Levi–Civita connection of g
Δ	The Laplace–Beltrami operator with respect to ∇, but sometimes also the usual Laplace operator
div	The covariant divergence operator acting on symmetric two–tensor fields or vector fields, but occasionally also the Euclidean divergence operator
Riem	The Riemann curvature tensor associated to g
Ric	The Ricci tensor obtained from Riem
Scal	The scalar curvature obtained from Ric
ν	The outer unit normal vector field to Σ in $\mathbb{R}^{n+1}$, also called *Gauss map*
A	The second fundamental form of Σ in $\mathbb{R}^{n+1}$
H	The mean curvature of Σ, $H = \mathrm{tr}_g A$
$\mathring{A}$	The traceless part of A, $\mathring{A} = A - \frac{H}{n} g$
$B{:}C$	The full contraction of the two smooth, symmetric two–tensor fields B and C, i.e., in coordinates, $B{:}C = \sum_{i,j,k,l=1}^n (g^{-1})^{ik}(g^{-1})^{jl} B_{ij} C_{kl}$

Also, for smooth functions $\varphi : \Sigma \to \mathbb{R}$, we set

$$\overline{\varphi} = \fint_\Sigma \varphi \, d\mathrm{vol}_g = \frac{1}{\mathrm{vol}_n(\Sigma)} \int_\Sigma \varphi \, d\mathrm{vol}_g$$

to denote the average of φ over the hypersurface Σ, but usually we do not specify the volume form when we write an integral.

Sign conventions. If X, Y and Z denote any smooth vector fields on Σ, extended to a neighbourhood of Σ, then we put

$$\mathrm{Riem}(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X - \nabla_X Y} Z,$$

and

$$A(X,Y) = -\big\langle D_X Y, \nu \big\rangle_{\mathbb{R}^{n+1}}.$$

These signs are chosen in such a way that the usual n–sphere S^n_R of radius $R > 0$ has scalar curvature $\mathrm{Scal}_{S^n_R} = \frac{n(n+1)}{R}$, and that the second fundamental form A of Σ has non–negative eigenvalues, whenever Σ bounds a convex domain.

CHAPTER 1

The super–critical case for generic hypersurfaces

In this chapter, we prove our main estimate for generic n–dimensional hypersurfaces of $\mathbb{R}^{n+1}$ in the case $p > n \geq 2$, and show that it implies qualitative C^0–closeness to a sphere. Unfortunately, the constant on the right–hand side of the estimate depends on the L^p–norm over the hypersurface of the second fundamental form of the hypersurface. At this point it is unclear to the author how to mend that.

Contents

1. The main theorem of this chapter

Our goal is to prove

Theorem 1.1. *Let $n \geq 2$, $p \in (n, +\infty)$ and $c_0 > 0$ be given. Then there is a constant $C > 0$, depending only on n, p and c_0, such that:*
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed and connected n–dimensional hypersurface with induced Riemannian metric g and such that

(a) $\mathrm{vol}_n(\Sigma) = 1$

and

(b) $\|A\|_{L^p(\Sigma)} = \left(\int_\Sigma |A|^p \right)^{\frac{1}{p}} \leq c_0,$

then

$$\min_{\lambda\in\mathbb{R}} \|A-\lambda g\|_{L^p(\Sigma)} \le C\,\|\mathring{A}\|_{L^p(\Sigma)}\,. \tag{1.1}$$

We prove the theorem in the next section, and show in Section 3 how it implies

Corollary 1.2 (to Theorem 1.1)**.** *Let $n \ge 2$, $p \in (n,+\infty)$, $c_0 > 0$ and $\epsilon > 0$ be given. Then there is a constant $\delta > 0$, depending only on n, p, c_0 and ϵ, such that: if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed and connected n–dimensional hypersurface such that*

(a) $\mathrm{vol}_n(\Sigma) = 1$,

(b) $\|A\|_{L^p(\Sigma)} \le c_0$

and

(c) $\|\mathring{A}\|_{L^p(\Sigma)} < \delta$,

then

$$d_{\mathrm{HD}}\big(\Sigma, \partial B_{\rho_0}(x)\big) < \epsilon, \qquad \textit{for some } x \in \mathbb{R}^{n+1},$$

where $\rho_0 = (\mathrm{vol}_n(S^n))^{-\frac{1}{n}}$ and d_{HD} denotes the Hausdorff distance in Euclidean $\mathbb{R}^{n+1}$.

The idea of the proof of Theorem 1.1 (performed in Section 2) is as follows. We will show that an estimate analogous to (1.1) holds in charts where a portion of Σ is given as the graph of a smooth function. This will be done by examining the differential equation that each component of A satisfies in such charts in terms of derivatives of $\mathring{A}$ and applying the Calderón–Zygmund inequality. In order to get the global estimate, we show that Σ can be covered by a controlled number of geodesic balls with a certain size and overlap and such that each ball is contained in the graph over a tangential plane of a smooth Lipschitz function. As will become clear in the proof, we have to assume an upper bound for $\|A\|_{L^p(\Sigma)}$ as well as $p > n$, so that we can "patch together" the local estimates to obtain the global one.

In order to get the local estimate, we will make use of the following, rather surprising result, which states that the partial derivatives (with respect to the Cartesian coordinates of a chart in which a portion of Σ is given as a graph) of the second fundamental form A are entirely determined by the partial derivatives of its traceless part $\mathring{A}$. The proof of this will be given in Section 4.

Lemma 1.3. *Let $U \subset \mathbb{R}^n$, $n \ge 2$, be an open set and assume Σ is the graph of a smooth function $u : U \to \mathbb{R}$ (Σ is thus a smooth hypersurface in $\mathbb{R}^{n+1}$). Let $\phi : U \to \mathbb{R}^{n+1}$, $x \mapsto \big(x, u(x)\big)$ be the corresponding parametrisation. Denote by D the derivation with respect to the Cartesian coordinates of $\mathbb{R}^n$ using the chart ϕ. Then the partial derivatives in U of the second fundamental form A of Σ satisfy*

$$D_k A^i{}_j = D_k \mathring{A}^i{}_j + \frac{1}{n-1}\left(\sum_{l=1}^{n} D_l \mathring{A}^l{}_k\right)\delta^i{}_j, \qquad \forall i,\, j,\, k, \tag{1.2}$$

where $\mathring{A}^i{}_j = A^i{}_j - \frac{1}{n}\sum_{l=1}^{n} A^l{}_l \delta^i{}_j$ are the components of the traceless part $\mathring{A}$ of A.

Remark 1.4. *Notice that the statement of Lemma 1.3 would be a rather obvious consequence of the Codazzi equations, if* (1.2) *were given with respect to the Levi–Civita connection* ∇. *The point here is, of course, that the identity holds for the usual "Euclidean partial derivatives" in the corresponding chart.*

We will think of expression (1.2) as a system of partial differential equations, where the unknowns are the components of A. The following proposition then yields the local estimate. Its proof, which is deferred to Section 5, is done by taking the trace of (1.2) to get an equation of the form $D\mathfrak{u} = \operatorname{div}\mathfrak{f}$ which, by the Calderón–Zygmund inequality, admits the desired estimate.

Proposition 1.5. *Let* $U \subset \mathbb{R}^n$, $n \geq 2$, *be an open set with* $0 \in U$, *and assume* Σ *is as in Lemma 1.3. Let* $R > 0$ *be such that* $D_R(0) \subset U$ *and assume* $p \in (1, +\infty)$ *be given. Then there exists a constant* $C > 0$, *which depends only on* n *and* p, *and there exists a* $\lambda \in \mathbb{R}$ *such that*

$$\|A - \lambda g\|_{L^p(D_{R/4}(0))} \leq C \, \|\mathring{A}\|_{L^p(D_R(0))} \, . \tag{1.3}$$

Remark 1.6. *Notice that, in contrast to* (1.1), *the constant* C *on the right–hand side of* (1.3) *is independent of the second fundamental form* A *of the hypersurface* Σ. *We will make use of this fact in the next chapter. Here, it is the lower bound on* R, *required to be able to apply Lemma 1.7 (see below), that will introduce this dependence. In addition, it is also this lower bound that will restrict the global theorem to the cases* $p > n$.

Now, to obtain the global estimate, we need the technical lemma below. It states that we can cover Σ by a controlled number of geodesic balls in which a portion of Σ is represented as a Lipschitz graph. As every smooth hypersurface can locally be parametrised as the graph of a Lipschitz map, the important assumption will be that there is a uniform upper bound on the Lipschitz constants of these maps, as well as a uniform lower bound on the size of the domain on which the maps are defined. As we will see in Section 2, the assumptions of Theorem 1.1 imply those of the lemma. The proof of the latter, performed in Section 6, is based on the observation that a geodesic sphere with small enough radius ρ is contained in the graph of one of the Lischitz maps over a ring with radii related to ρ. Consequently, the volume of small geodesic balls on Σ is controlled by the volume of Euclidean balls. Since Σ has normalised area and is compact, the existence of the aforementioned cover is then assured.

Lemma 1.7. *Let* $\Sigma \subset \mathbb{R}^{n+1}$, $n \geq 2$, *be a smooth closed hypersurface with normalised area,* $\operatorname{vol}_n(\Sigma) = 1$, *and let* $r_0 > 0$ *and* $L > 0$ *be given. Assume that, for each point* $q \in \Sigma$, *there is an isometry* Φ_q *of* $\mathbb{R}^{n+1}$ *and a smooth Lipschitz function* $u_q : D_{r_0}(0) \subset \mathbb{R}^n \to \mathbb{R}$ *with Lipschitz constant at most* L, *such that* $\Phi_q\Big(D_{r_0}(0), u_q\big(D_{r_0}(0)\big)\Big) \subset \Sigma$ *and* $\Phi_q\big(0, u_q(0)\big) = q$.
Then, for every $q \in \Sigma$, *the geodesic ball* $\mathcal{B}^g_{r_0}(q) \subset \Sigma$ *of radius* r_0 *around* q *is contained in* $\Phi_q\Big(D_{r_0}(0), u_q\big(D_{r_0}(0)\big)\Big)$, *and there is a constant* C, *depending only on* n, *such*

that Σ can be covered with N such geodesic balls, where

$$N \leq C\frac{(1+L)^{2n}}{r_0^n}. \tag{1.4}$$

2. Proof of Theorem 1.1

In this section we deduce the global rigidity estimate (1.1) from the local rigidity estimate (Proposition 1.5) and the existence of a covering as described in Lemma 1.7.

2.1. Construction of Lipschitz charts. So let Σ be as in Theorem 1.1 and pick any point $q \in \Sigma$. Without loss of generality, we may assume that $q = 0 \in \mathbb{R}^{n+1}$ and $T_q\Sigma = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. By smoothness, a portion of Σ is then given as the graph of a smooth function $u : U \subset \mathbb{R}^n \to \mathbb{R}$ with $u(0) = 0$ and $Du(0) = 0$, where D denotes derivation with respect to the Cartesian coordinates of $\mathbb{R}^n$. We can assume that U is maximal, in the sense that if a portion of Σ can be represented as a graph over $V \supset U$, then $V = U$ necessarily.

In the proof of Lemma 1.3 (Section 4), we will see that, in the coordinates at hand, the metric g of Σ, its inverse and the second fundamental form A of Σ are given by (see equations (1.13), (1.14) and (1.15)):

$$g_{ij} = \delta_{ij} + D_iuD_ju, \qquad g^{ij} = \delta^{ij} - \frac{D^iuD^ju}{1+|Du|^2}, \qquad A_{ij} = \frac{D_iD_ju}{\sqrt{1+|Du|^2}},$$

respectively. Let

$$v = \frac{Du}{\sqrt{1+|Du|^2}}.$$

Notice that, by definition, $|v| < 1$. Moreover, if $0 < c < 1$ and $|v| \leq c$ then $|Du| \leq c/\sqrt{1-c^2} < +\infty$, implying that u is Lipschitz with Lipschitz constant $L \leq c/\sqrt{1-c^2}$.

Define

$$R = \sup\left\{ r > 0 \;\middle|\; \sup_{e \in \partial D_1(0)} |v(re)| \leq \frac{1}{2} \right\}$$

as the maximal radius of a ball $D_R(0) \subset \mathbb{R}^n$ such that the length of v is uniformly bounded by $1/2$. Clearly, $R > 0$ since $v(0) = 0$ and v is continuous. Also, $R < +\infty$ since otherwise u would be defined on the whole of $\mathbb{R}^n$ and Σ would not be compact. In fact, in view of equation (1.13) of Section 4 and our assumption that $\mathrm{vol}_n(\Sigma) = 1$, we must have $R \leq \left(\mathrm{vol}_n(S^n)\right)^{-\frac{1}{n}}$. Thus R is well–defined and u is at least defined on $D_R(0)$. Moreover $|v| \leq 1/2$ holds throughout $D_R(0)$, implying that u is uniformly Lipschitz on $D_R(0)$ with constant $1/\sqrt{3}$.

2.2. The local estimate. Applying Proposition 1.5 for $p > n$, we obtain a constant C, depending only on n and p, and some $\lambda \in \mathbb{R}$ such that

$$\|A - \lambda g\|_{L^p(D_{r/4}(0))} \leq C \,\|\mathring{A}\|_{L^p(D_r(0))}, \qquad \forall r \leq R.$$

Remark 1.8. *As we will see in the proof of Lemma 1.7 (Section 6), any geodesic ball $\mathcal{B}^g_r(q)$ with centre q and radius $r \le R$ is contained in the graph of u over $D_r(0)$. Moreover, since u is Lipschitz, the area* $\mathrm{vol}_n(\mathcal{B}^g_r(q))$ *of such a geodesic ball is controlled by* $\mathrm{vol}_n(D_r(0))$*. This will be useful later.*

The first part of the remark yields the following local estimate: For all $q \in \Sigma$ there is a $\lambda \in \mathbb{R}$ such that for $r \le R/4$

$$\|A - \lambda g\|_{L^p(\mathcal{B}^g_r(q))} \le C \,\|\mathring{A}\|_{L^p(\Sigma)}\,, \tag{1.5}$$

where C depends only on n and p.

2.3. A lower bound on the size of the Lipschitz charts. In order to apply Lemma 1.7 to get the global analogue of the above estimate, we now show that R is bounded from below. For this we calculate:

$$\begin{aligned} D_j v^i &= D_j \frac{D^i u}{\sqrt{1+|Du|^2}} = \frac{D_j D^i u}{\sqrt{1+|Du|^2}} - \frac{\sum_{l=1}^n D^i u D^l u D_j D_l u}{(1+|Du|)^{3/2}} \\ &= \sum_{l=1}^n \left(\delta^{il} - \frac{D^i u D^l u}{1+|Du|^2}\right) \frac{D_l D_j u}{\sqrt{1+|Du|^2}} = \sum_{l=1}^n g^{il} A_{lj} = A^i{}_j. \end{aligned} \tag{1.6}$$

Thus $|Dv| = |A|$ in every point of $D_R(0)$. We apply the Morrey–type estimate found in Lemma A.1 of the appendix to v at $x = 0$, and we get with identity (1.6)

$$\sup_{y \in D_R(0)} \frac{|v(y)|}{|y|^{(p-n)/p}} \le C \,\|Dv\|_{L^p(D_R(0))} = C \,\|A\|_{L^p(D_R(0))}\,.$$

Now, by the maximality of R, there exists an $e \in \partial D_1(0)$ such that $\lim_{r \nearrow R} |v(re)| = 1/2$. For $y = Re$, we therefore obtain

$$\frac{1/2}{R^{(p-n)/p}} \le C \,\|A\|_{L^p(D_R(0))}\,. \tag{1.7}$$

Since $\|A\|_{L^p(\Sigma)} \le c_0$ by assumption, we infer that there is a constant C, depending only on n and p, such that, if we define

$$R_0 = C c_0^{-p/(p-n)}, \tag{1.8}$$

then $R \ge R_0$.

Remark 1.9. *Conversely, in view of the upper bound* $\left(\mathrm{vol}_n(S^n)\right)^{-\frac{1}{n}}$ *on R coming from the assumption that* $\mathrm{vol}_n(\Sigma) = 1$ *(see* (1.13) *in Section 4), we infer from inequality* (1.7) *that*

$$\|A\|_{L^p(\Sigma)} \ge \frac{\left(\mathrm{vol}_n(S^n)\right)^{\frac{n-p}{np}}}{2C'},$$

where C' is the constant of Lemma A.1 that depends only on n and p. This observation will be useful in the proof of the qualitative C^0–closeness in Section 3.

2.4. Local to global. Taking $r_0 = R_0/8$ and $L = 1/\sqrt{3}$, we can now apply Lemma 1.7 (twice — once both statements, then only the first) to obtain a covering $\{\mathcal{B}^g_{r_0}(q_j)\}_{1\leq j\leq N}$ of Σ by geodesic balls of radius r_0 such that, in each $\mathcal{B}^g_{2r_0}(q_j)$, the local estimate (1.5) holds for all $r \leq 2r_0 = R_0/4$. By the triangle inequality, any two balls of the covering that intersect will have the property that the balls with same centres but twice the radius have an overlap that contains, at least, a geodesic ball of radius r_0. This will be useful to "patch together" the local estimates in order to obtain the global one, since, obviously, λ depends on the geodesic ball (cf. the proof of Proposition 1.5 in Section 5). Indeed, given that the covering of Σ by $\left\{\mathcal{B}^g_{2r_0}(q_j)\right\}_{1\leq j\leq N}$ has sufficiently large overlaps, the difference of the λs in two neighbouring balls is controlled.

In fact, let $\Omega_1, \Omega_2 \subset \Sigma$ with $\mathrm{vol}_n(\Omega_1 \cap \Omega_2) > 0$ and assume that there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\|A - \lambda_1 g\|_{L^p(\Omega_1)} \leq \beta$ and $\|A - \lambda_2 g\|_{L^p(\Omega_2)} \leq \beta$ for some β independent of Ω_1 and Ω_2. We have

$$
\begin{aligned}
|\lambda_1 - \lambda_2| = (|\lambda_1 - \lambda_2|^p)^{\frac{1}{p}} &= \left(\frac{1}{\mathrm{vol}_n(\Omega_1\cap\Omega_2)}\int_{\Omega_1\cap\Omega_2} |\lambda_1 - \lambda_2|^p\right)^{\frac{1}{p}} \\
&= \frac{1}{\mathrm{vol}_n(\Omega_1\cap\Omega_2)^{\frac{1}{p}}} \|\lambda_1 - \lambda_2\|_{L^p(\Omega_1\cap\Omega_2)} \\
&= \frac{1}{n\mathrm{vol}_n(\Omega_1\cap\Omega_2)^{\frac{1}{p}}} \|\lambda_1 g - A + A - \lambda_2 g\|_{L^p(\Omega_1\cap\Omega_2)} \\
&\leq \frac{1}{n\mathrm{vol}_n(\Omega_1\cap\Omega_2)^{\frac{1}{p}}} \left(\|A - \lambda_1 g\|_{L^p(\Omega_1\cap\Omega_2)} + \|A - \lambda_2 g\|_{L^p(\Omega_1\cap\Omega_2)}\right) \\
&\leq \frac{2\beta}{n\mathrm{vol}_n(\Omega_1\cap\Omega_2)^{\frac{1}{p}}}.
\end{aligned}
$$

In particular, if Ω_1 and Ω_2 are two intersecting geodesic balls of radius r_0 from the cover, such that the intersection of the balls with doubled radius contains a geodesic ball of radius r_0, then the local estimate (1.5), together with Remark 1.8, yields a constant C, depending only on n and p, such that

$$
|\lambda_1 - \lambda_2| \leq C r_0^{-\frac{n}{p}} \|\mathring{A}\|_{L^p(\Sigma)}.
$$

Consider a path joining the ball in the cover with the smallest λ, say $\lambda_{\min}$, to the one with the largest λ, say $\lambda_{\max}$. Since the path can cross at most N distinct balls, we find that

$$
|\lambda_{\max} - \lambda_{\min}| \leq C r_0^{-\frac{n}{p}} N \|\mathring{A}\|_{L^p(\Sigma)}, \tag{1.9}
$$

where the constant C depends only on n and p. Let $\mathcal{B}^g_j$, $j = 1, \ldots, N$, denote the geodesic balls of the cover and λ_j their corresponding λs. By virtue of (1.9) above and the local estimate (1.5) we then have for any λ between $\lambda_{\min}$ and $\lambda_{\max}$,

$$\begin{aligned}
\|A-\lambda g\|_{L^p(\Sigma)} &\leq \sum_{j=1}^{N} \|A-\lambda g\|_{L^p(\mathcal{B}_j^g)} \\
&= \sum_{j=1}^{N} \|A-\lambda_j g+\lambda_j g-\lambda g\|_{L^p(\mathcal{B}_j^g)} \\
&\leq \sum_{j=1}^{N} \Big(\|A-\lambda_j g\|_{L^p(\mathcal{B}_j^g)} + \|\lambda_j g-\lambda g\|_{L^p(\mathcal{B}_j^g)} \Big) \\
&\leq \sum_{j=1}^{N} \Big(\|A-\lambda_j g\|_{L^p(\mathcal{B}_j^g)} + n\,|\lambda_{\max}-\lambda_{\min}|\,\mathrm{vol}_n(\mathcal{B}_j^g)^{\frac{1}{p}} \Big) \\
&\leq \sum_{j=1}^{N} C_1\,\|\mathring{A}\|_{L^p(\Sigma)} \Big(1+C_2 N r_0^{-\frac{n}{p}} (r_0^n)^{\frac{1}{p}} \Big) \\
&\leq \sum_{j=1}^{N} C_3(1+N)\,\|\mathring{A}\|_{L^p(\Sigma)} \\
&\leq CN^2\,\|\mathring{A}\|_{L^p(\Sigma)}\,,
\end{aligned}$$

where the constants C_1, C_2, C_3 and C depend only on n and p. Using the upper bound (1.4) on the number N of balls in the cover and the expression (1.8) for $4r_0$ we finally obtain

$$\|A-\lambda g\|_{L^p(\Sigma)} \leq C c_0^{\frac{2np}{p-n}}\,\|\mathring{A}\|_{L^p(\Sigma)}\,,$$

where, again, C depends only on n and p. This proves Theorem 1.1. □

3. Proof of Corollary 1.2

In this section we want to show that any hypersurface Σ that fulfils the assumptions of Theorem 1.1 has to be C^0–close to a sphere, whenever $\|\mathring{A}\|_{L^p(\Sigma)}$ is small enough. We do this through a contradiction argument.

3.1. Preliminaries. Assume Corollary 1.2 were false. Then we would find a sequence $(\Sigma_k)_{k\in\mathbb{N}}$ of smooth, closed and connected hypersurfaces of $\mathbb{R}^{n+1}$, satisfying $\mathrm{vol}_n(\Sigma_k)=1$ and $\|A\|_{L^p(\Sigma_k)} \leq c_0$ independently of k, and such that

$$\lim_{k\to\infty} \|\mathring{A}\|_{L^p(\Sigma_k)} = 0,$$

and the hypersurfaces Σ_k do *not* converge (in the Hausdorff topology) to a ball (notice that we do not even claim that there is a limit set). We shall show that this is impossible.

We begin with the following observations. Applying Theorem 1.1 to each Σ_k, we get, for every $k\in\mathbb{N}$, a $\lambda_k\in\mathbb{R}$ such that

$$\|A-\lambda_k g\|_{L^p(\Sigma_k)} \leq C\,\|\mathring{A}\|_{L^p(\Sigma_k)}\,,$$

where $C > 0$ depends only on n, p and c_0. The sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$, since

$$
\begin{aligned}
|\lambda_k| &= \frac{1}{\sqrt{n}}\,\|\lambda_k g\|_{L^p(\Sigma_k)} \leq \frac{1}{\sqrt{n}}\,\|A\|_{L^p(\Sigma_k)} + \frac{1}{\sqrt{n}}\,\|A-\lambda_k g\|_{L^p(\Sigma_k)} \\
&\leq \frac{1}{\sqrt{n}}\,\|A\|_{L^p(\Sigma_k)} + \frac{C}{\sqrt{n}}\,\|\mathring{A}\|_{L^p(\Sigma_k)} \leq \frac{c_0}{\sqrt{n}} + \frac{C}{\sqrt{n}}\,\|\mathring{A}\|_{L^p(\Sigma_k)}\,,
\end{aligned}
\tag{1.10}
$$

and the second term on the right–hand side converges to zero as $k \to \infty$. Hence, modulo picking a subsequence, we might without loss of generality assume that $\lim_{k\to\infty}\lambda_k = \overline{\lambda} \in \mathbb{R}$. Notice also that, in view of Remark 1.9 in the proof of Theorem 1.1 (Section 2), we have for each k, that $\|A\|_{L^p(\Sigma_k)} \geq \delta$, where $\delta > 0$ depends only on n and p. It follows that

$$
\begin{aligned}
|\lambda_k| &= \frac{1}{\sqrt{n}}\,\|\lambda_k g\|_{L^p(\Sigma_k)} \geq \frac{1}{\sqrt{n}}\,\|A\|_{L^p(\Sigma_k)} - \frac{1}{\sqrt{n}}\,\|A-\lambda_k g\|_{L^p(\Sigma_k)} \\
&\geq \frac{\delta}{\sqrt{n}} - \frac{C}{\sqrt{n}}\,\|\mathring{A}\|_{L^p(\Sigma_k)}\,,
\end{aligned}
\tag{1.11}
$$

whence $\left|\overline{\lambda}\right| \geq \delta/\sqrt{n} > 0$.

Returning to the main argument, we show how our assumptions imply that, locally, the Σ_ks have to converge to portions of spheres.

3.2. Local convergence. We pick, for each $k \in \mathbb{N}$, an arbitrary point $q_k \in \Sigma_k$. Modulo translations and rotations, we can without loss of generality assume that $q_k = 0 \in \mathbb{R}^{n+1}$, and that $T_{q_k}\Sigma_k = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ (cf. the proof of Theorem 1.1). Then, from Section 2, we know that each Σ_k has a portion given as the graph of a smooth $1/\sqrt{3}$–Lipschitz function $u_k : \overline{D_R(0)} \to \mathbb{R}$, where R depends only on n, p and c_0 (notice that, by our construction in the proof of Theorem 1.1, the functions u_k are Lipschitz up to the boundary of $D_R(0) \subset \mathbb{R}^n$). The sequence $(u_k)_{k\in\mathbb{N}}$ is therefore a pointwise bounded, equicontinuous sequence in the space of continuous, real–valued functions on the compact domain $\overline{D_R(0)}$. Then the Ascoli–Arzelà–Theorem (see, e.g., [**Rud91**, Thm.A5, p.394]) implies the existence of a subsequence $\left(u_{k_l}\right)_{l\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ that converges uniformly on $\overline{D_R(0)}$ to a continuous function $\overline{u}$.

Now define for each l, as in the proof of Theorem 1.1,

$$
v_l = \frac{Du_{k_l}}{\sqrt{1+|Du_{k_l}|^2}}.
$$

There, we had seen that $D_j(v_l)^i = (A_{u_{k_l}})^i{}_j$ for all $i, j \in \{1,\dots,n\}$ (see eq. (1.6)). Then, since $|Du_{k_l}| \leq \frac{1}{\sqrt{3}}$ (implying that $|v_l| \leq \frac{1}{2}$) and, using our bound on the L^p–norm of A,

$$
\begin{aligned}
\|Dv_l\|_{L^1(\overline{D_R(0)})} &\leq \left(\mathrm{vol}_n\left(\overline{D_R(0)}\right)\right)^{1-\frac{1}{p}} \left\|A_{u_{k_l}}\right\|_{L^p(\overline{D_R(0)})} \\
&\leq c_0 \left(\mathrm{vol}_n\left(\overline{D_R(0)}\right)\right)^{1-\frac{1}{p}}\,,
\end{aligned}
$$

$(v_l)_{l\in\mathbb{N}}$ is bounded in $W^{1,1}(\overline{D_R(0)};\mathbb{R}^n)$. Consequently, by Rellich–Kondrachov (see, e.g, [**Eva98**, Thm.1, §5.7, p.272]), there is a subsequence $\left(v_{l_m}\right)_{m\in\mathbb{N}} \subset (v_l)_{l\in\mathbb{N}}$ and a (vector–valued) function $\overline{v} \in L^1(\overline{D_R(0)};\mathbb{R}^n)$ to which the v_{l_m} converge in L^1.

For each $m \in \mathbb{N}$ and $y \in \overline{D_R(0)}$, let $w_m(y) = v_{l_m}(y) - \overline{\lambda} y$. Then the w_m converge in L^1 to $\overline{w} = (\overline{v} - \overline{\lambda}\,\cdot)$. But thanks to Theorem 1.1, we also have

$$\begin{aligned}
\|Dw_m\|_{L^1(\overline{D_R(0)})} &= \left\|Dv_{l_m} - \overline{\lambda}\,\mathrm{id}\right\|_{L^1(\overline{D_R(0)})} \\
&\leq \left(\mathrm{vol}_n\left(\overline{D_R(0)}\right)\right)^{1-\frac{1}{p}} \left\|A_{u_{k_{l_m}}} - \overline{\lambda} g_{u_{k_{l_m}}}\right\|_{L^p(\overline{D_R(0)})} \\
&\leq \left(\mathrm{vol}_n\left(\overline{D_R(0)}\right)\right)^{1-\frac{1}{p}} \left(\|\mathring{A}\|_{L^p(\Sigma_{k_{l_m}})} + \left\|\lambda_{k_{l_m}} g - \overline{\lambda} g\right\|_{L^p(\Sigma_{k_{l_m}})}\right) \\
&\overset{m\to\infty}{\longrightarrow} \; 0.
\end{aligned}$$

It follows, that $D\overline{w} = 0$ in the sense of distributions (see, e.g., [**GT01**, Thm.7.4, p.150]), implying that $\overline{w} = c$ almost everywhere, for some $c \in \mathbb{R}^n$ (see, e.g., [**LL97**, Thm.6.11, p.138]). By the L^1–convergence of the v_{l_m} to $\overline{v}$, we may then, after picking a subsequence, assume without loss of generality that for almost every $y \in \overline{D_R(0)}$,

$$v_{l_m}(y) = \frac{Du_{k_{l_m}}(y)}{\sqrt{1 + \left|Du_{k_{l_m}}(y)\right|^2}} \quad \overset{m\to\infty}{\longrightarrow} \quad \overline{\lambda} y + c. \tag{1.12}$$

Since all the v_{l_m} are bounded in modulus by $\frac{1}{2}$, we first observe that $\left|\overline{\lambda} y + c\right| \leq \frac{1}{2}$ for all $y \in \overline{D_R(0)}$, necessarily. Moreover, since the map $z \mapsto \frac{z}{\sqrt{1-|z|^2}}$ is (even uniformly) continuous on $\overline{D_{\frac{1}{2}}(0)} \subset D_1(0)$, it follows that

$$Du_{k_{l_m}}(y) \quad \overset{m\to\infty}{\longrightarrow} \quad \frac{\overline{\lambda} y + c}{\sqrt{1 - |\overline{\lambda} y + c|^2}}, \qquad \text{for almost every } y \in \overline{D_R(0)}.$$

Since the $Du_{k_{l_m}}$ are uniformly bounded, the dominated convergence theorem (see, e.g., [**Rud87**, Thm.1.34, p.26]) then yields that the above convergence also holds in $L^1(\overline{D_R(0)})$. In addition, the $u_{k_{l_m}}$ themselves obviously converge in L^1 to $\overline{u}$, as well. Consequently, as before, we conclude that

$$D\overline{u} = \frac{\overline{\lambda}\cdot + c}{\sqrt{1 - |\overline{\lambda}\cdot + c|^2}}$$

in the sense of distributions. But then we argue once more that there must be constant $b \in \mathbb{R}$ such that, for almost every $y \in \overline{D_R(0)}$,

$$\overline{u}(y) = b - \sqrt{1 - |\overline{\lambda} y + c|^2}.$$

Since we established already that $\overline{u}$ is continuous, the above identity holds in all of $\overline{D_R(0)}$. Thus, indeed, $\overline{u}$ parametrises a portion of a sphere of radius $|\overline{\lambda}|^{-1}$ and centre $\left(-\frac{c}{\overline{\lambda}}, b\right)$.

We now show how we can easily obtain the global statement from Lemma 1.7.

3.3. Local to global. Applying the same technique as in the proof of Theorem 1.1 (Section 2), we cover each $\Sigma_{k_{l_m}}$ with geodesic balls of radius $2r_0 = R/4$ such that neighbouring balls have large enough overlap. Then our local argument shows that the centres of the two spherical portions to which $\left(\Sigma_{k_{l_m}}\right)_{m\in\mathbb{N}}$ converges in neighbouring balls have to coincide. We conclude immediately that the whole sequence has to converge to a sphere of radius $|\overline{\lambda}|^{-1}$, which contradicts our assumptions. This finishes the proof of the corollary. □

4. Proof of Lemma 1.3

Let $\Sigma \subset \mathbb{R}^{n+1}$ be as in Lemma 1.3. Σ is embedded into $\mathbb{R}^{n+1}$ by the map

$$f : U \to \mathbb{R}^{n+1}, \quad x \mapsto f^\alpha(x) = \begin{cases} x^\alpha, & \alpha \in \{1, \ldots, n\}, \\ u(x), & \alpha = n+1. \end{cases}$$

The metric of Σ in the given coordinates x^i is obtained from $g_{ij} = \sum_{\alpha=1}^{n+1} D_i f^\alpha D_j f_\alpha$. We adopt the convention that Greek indices run from 1 to $n+1$ (representing coordinates in the ambient space $\mathbb{R}^{n+1}$), whereas Latin ones run from 1 to n (representing coordinates in the coordinate space $\mathbb{R}^n$). Since

$$D_i f^\alpha = \begin{cases} \delta_i{}^\alpha, & \alpha \in \{1, \ldots, n\}, \\ D_i u, & \alpha = n+1, \end{cases}$$

we get

$$g_{ij} = \sum_{k=1}^{n} \delta_i{}^k \delta_{kj} + D_i u D_j u = \delta_{ij} + D_i u D_j u. \tag{1.13}$$

It is easy to verify that the inverse of g is then given by

$$g^{ij} = \delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}, \tag{1.14}$$

where $|Du|^2 = \sum_{i=1}^{n} D^i u D_i u$.

Remark 1.10. *In the above expressions* (1.13) *and* (1.14)*, the indices on the right–hand side refer to the coordinates in* $\mathbb{R}^{n+1}$*. Therefore, they are raised and lowered with the ambient metric* δ*. On the left–hand side, however, the indices are with respect to the coordinate space* $\mathbb{R}^n$*. Those indices are raised and lowered with* g*.*

The Gauss map of Σ is given by

$$\nu^\alpha = \frac{1}{\sqrt{1 + |Du|^2}} \begin{cases} D^\alpha u, & \alpha \in \{1, \ldots, n\}, \\ -1, & \alpha = n+1, \end{cases}$$

where we have chosen the orientation such that $\det(Df,\nu) < 0$. Consequently,

$$D_i\nu^\alpha = \begin{cases} \dfrac{D_iD^\alpha u}{\sqrt{1+|Du|^2}} + \dfrac{-D^\alpha u\left(\sum_{k=1}^n\left((D_iD^ku)D_ku + D^ku(D_iD_ku)\right)\right)}{2(1+|Du|^2)^{3/2}}, & \alpha\in\{1,\dots,n\},\\[2ex] \dfrac{2\sum_{k=1}^n D^ku(D_iD_ku)}{2(1+|Du|^2)^{3/2}}, & \alpha = n+1,\end{cases}$$

from which we calculate the second fundamental form A of Σ:

$$A_{ij} = \sum_{\alpha=1}^{n+1} D_i\nu^\alpha D_jf_\alpha = \frac{D_iD_ju}{\sqrt{1+|Du|^2}}. \tag{1.15}$$

To prove (1.2), we first calculate the Christoffel symbols of g:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^n g^{kl}\left(D_ig_{jl} + D_jg_{il} - D_lg_{ij}\right) = \frac{D^kuD_iD_ju}{1+|Du|^2} = v^kA_{ij},$$

where $v = \frac{Du}{\sqrt{1+|Du|^2}}$. Note that v is the projection of the Gauss map ν onto the tangent plane to U. Denoting by ∇ the Levi–Civita connection of g, we have for any indices i, j and k

$$\nabla_kA^i{}_j = D_kA^i{}_j + \sum_{l=1}^n\Gamma^i_{kl}A^l{}_j - \sum_{l=1}^n\Gamma^l_{kj}A^i{}_l.$$

Using the Codazzi equations,

$$\nabla_kA^i{}_j = \nabla_jA^i{}_k,$$

we obtain

$$D_kA^i{}_j = D_jA^i{}_k - \sum_{l=1}^n\Gamma^i_{kl}A^l{}_j + \sum_{l=1}^n\Gamma^i_{jl}A^l{}_k,$$

which, inserting the expression for the Christoffel symbols, reads

$$\begin{aligned} D_kA^i{}_j &= D_jA^i{}_k - v^i\sum_{l,s=1}^n A_{kl}g^{ls}A_{sj} + v^i\sum_{l,t=1}^n A_{jl}g^{lt}A_{tk}\\ &= D_jA^i{}_k. \end{aligned}$$

Now denote by $\mathring{A}$ the traceless part of A, i.e. $\mathring{A}^i{}_j = A^i{}_j - \frac{1}{n}\sum_{k=1}^n A^k{}_k\delta^i{}_j$. Clearly, if $i\neq j$, we have for all k

$$D_kA^i{}_j = D_k\mathring{A}^i{}_j.$$

Consequently, we have for all $k\neq i$

$$D_kA^i{}_i = D_iA^i{}_k = D_i\mathring{A}^i{}_k.$$

We finally calculate

$$\begin{aligned}
D_i A^i{}_i &= D_i \mathring{A}^i{}_i + \frac{1}{n} \sum_{k=1}^{n} D_i A^k{}_k \\
&= D_i \mathring{A}^i{}_i + \frac{1}{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} D_i A^k{}_k + \frac{1}{n} D_i A^i{}_i \\
&= \frac{1}{1-\frac{1}{n}} \left(D_i \mathring{A}^i{}_i + \frac{1}{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} D_k \mathring{A}^k{}_i \right) \\
&= \frac{n}{n-1} \left(\left(1 - \frac{1}{n}\right) D_i \mathring{A}^i{}_i + \frac{1}{n} \sum_{k=1}^{n} D_k \mathring{A}^k{}_i \right) \\
&= D_i \mathring{A}^i{}_i + \frac{1}{n-1} \sum_{k=1}^{n} D_k \mathring{A}^k{}_i .
\end{aligned}$$

This yields for arbitrary indices i, j and k:

$$\begin{aligned}
D_k A^i{}_j &= D_k \mathring{A}^i{}_j + \frac{1}{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} D_k A^l{}_l \delta^i{}_j + \frac{1}{n} D_k A^k{}_k \delta^i{}_j \\
&= D_k \mathring{A}^i{}_j + \frac{1}{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} D_l \mathring{A}^l{}_k \delta^i{}_j + \frac{1}{n} D_k \mathring{A}^k{}_k \delta^i{}_j + \frac{1}{n(n-1)} \sum_{l=1}^{n} D_l \mathring{A}^l{}_k \delta^i{}_j \\
&= D_k \mathring{A}^i{}_j + \frac{1}{n} \sum_{l=1}^{n} D_l \mathring{A}^l{}_k \delta^i{}_j + \frac{1}{n(n-1)} \sum_{l=1}^{n} D_l \mathring{A}^l{}_k \delta^i{}_j \\
&= D_k \mathring{A}^i{}_j + \frac{1}{n-1} \sum_{l=1}^{n} D_l \mathring{A}^l{}_k \delta^i{}_j .
\end{aligned}$$

This proves (1.2) and thus Lemma 1.3. □

5. Proof of Proposition 1.5

Proposition 1.5 will follow directly from the following

Proposition 1.11. *Let $n \geq 2$, $R > 0$ and $1 < p < +\infty$. Let $\mathfrak{u} \in C^3(B_R(0)) \cap L^p(B_R(0))$ and $\mathfrak{f} \in C^3(B_R(0); \mathbb{R}^{n\times n}) \cap L^p(B_R(0); \mathbb{R}^{n\times n})$ be such that $\mathfrak{u}$ solves on $B_R(0) \subset \mathbb{R}^n$*

$$D\mathfrak{u} = \operatorname{div} \mathfrak{f}, \tag{1.16}$$

i.e.

$$D_i \mathfrak{u} = \sum_{k=1}^{n} D_k \mathfrak{f}^k{}_i .$$

Then there is a constant C, depending only on n and p, such that

$$\left\| \mathfrak{u} - \fint_{B_{R/4}(0)} \mathfrak{u} \right\|_{L^p(B_{R/4}(0))} \leq C \left\| \mathfrak{f} \right\|_{L^p(B_R(0))}, \tag{1.17}$$

where $\fint_{B_{R/4}(0)} \mathfrak{u} = \mathrm{vol}_n(B_{R/4}(0))^{-1} \int_{B_{R/4}(0)} \mathfrak{u}$ *denotes the average of* $\mathfrak{u}$ *on* $B_{R/4}(0)$.

Indeed, Proposition 1.11 implies Proposition 1.5 if we take

$$\mathfrak{u} = \sum_{k=1}^{n} {A^k}_k \qquad \text{and} \qquad \mathfrak{f} = \frac{n}{n-1} \mathring{A}.$$

For then, by equation (1.2) of Lemma 1.3, $\mathfrak{u}$ satisfies $D\mathfrak{u} = \operatorname{div} f$, and (1.17) yields

$$\left\| \sum_{k=0}^{n} {A^k}_k - \fint_{D_{R/4}(0)} \sum_{k=0}^{n} {A^k}_k \right\|_{L^p(D_{R/4}(0))} \leq C \left\| \mathring{A} \right\|_{L^p(D_R(0))},$$

for some constant C that depends only on n and p. Writing

$$\lambda = \frac{1}{n} \fint_{D_{R/4}(0)} \sum_{k=1}^{n} {A^k}_k,$$

we obtain

$$\begin{aligned} \left\| A - \lambda g \right\|_{L^p(D_{R/4}(0))} &= \left\| \mathring{A} + \frac{1}{n} \left(\sum_{k=1}^{n} {A^k}_k \right) g - \lambda g \right\|_{L^p(D_{R/4}(0))} \\ &\leq \left\| \mathring{A} \right\|_{L^p(D_{R/4}(0))} + \frac{n}{n} \left\| \sum_{k=1}^{n} {A^k}_k - \fint_{D_{R/4}(0)} \sum_{k=1}^{n} {A^k}_k \right\|_{L^p(D_{R/4}(0))} \\ &\leq C \left\| \mathring{A} \right\|_{L^p(D_R(0))}, \end{aligned}$$

which proves Proposition 1.5. □

Proof of Proposition 1.11. Let $\varphi \in C^\infty(\mathbb{R}; [0,1])$ be a smooth function with the following properties:

(i) $\forall x \in \mathbb{R}$, $\varphi(-x) = \varphi(x)$,
(ii) $\forall |x| \leq 1/2$, $\varphi(x) = 1$,
(iii) $\forall |x| \geq 1$, $\varphi(x) = 0$,

and

(iv) If $|x| = 1$, then $\varphi(x) = \varphi'(x) = \varphi''(x) = 0$.

Define

$$\widetilde{\mathfrak{f}}(x) = \begin{cases} \mathfrak{f}(x) \varphi\left(\frac{|x|}{R}\right), & |x| < R, \\ 0, & |x| \geq R. \end{cases}$$

Then $\|\widetilde{\mathfrak{f}}\|_{L^p(\mathbb{R}^n)} \leq \|\mathfrak{f}\|_{L^p(B_R(0))}$ and $\widetilde{\mathfrak{f}} = D_i\widetilde{\mathfrak{f}} = 0$, $\forall i$, on $\partial B_R(0)$. Moreover, $\mathfrak{u}$ solves $D\mathfrak{u} = \operatorname{div}\widetilde{\mathfrak{f}}$ in $B_{R/2}(0)$. Let $\mathfrak{w}$ be the fundamental solution of

$$\begin{cases} \Delta\mathfrak{w} = \operatorname{div}\operatorname{div}\widetilde{\mathfrak{f}}, & \text{in } B_R(0), \\ \quad \mathfrak{w} = 0, & \text{on } \partial B_R(0), \end{cases}$$

and denote by K the standard Dirichlet kernel in $\mathbb{R}^n$. Then $\mathfrak{w}$ is given by

$$\begin{aligned} \mathfrak{w} &= \int_{B_R(0)} K(y-x)\operatorname{div}_y\operatorname{div}_y\widetilde{\mathfrak{f}}(y)\,dy \\ &= \int_{B_R(0)} \sum_{k,l=1}^n K(y-x) D_y^k D_y^l \widetilde{\mathfrak{f}}^k{}_l(y)\,dy \\ &= \int_{B_R(0)} \sum_{k,l=1}^n D_y^k D_y^l K(y-x) \widetilde{\mathfrak{f}}^k{}_l(y)\,dy \\ &= \int_{\mathbb{R}^n} \sum_{k,l=1}^n D_y^k D_y^l K(y-x) \widetilde{\mathfrak{f}}^k{}_l(y)\,dy \\ &= \sum_{k,l=1}^n \frac{1}{n\widetilde{\omega}_n} \int_{\mathbb{R}^n} \frac{n(y-x)^k(y-x)^l - |y-x|^2\,\delta^{kl}}{|y-x|^{n+2}} \widetilde{\mathfrak{f}}^k{}_l(y)\,dy, \end{aligned}$$

where $\widetilde{\omega}_n = \operatorname{vol}_n(B_1(0))$ denotes the volume of the unit ball in $\mathbb{R}^n$. It is straightforward to check that, for any k and l, the map

$$\Omega^{kl} : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \frac{nx^k x^l - |x|^2\,\delta^{kl}}{|x|^2},$$

which is homogeneous of degree 0, satisfies the cancellation property,

$$\int_{\partial B_1(0)} \Omega^{kl} = 0,$$

and the smoothness condition,

$$\int_0^1 \frac{\sup\limits_{\substack{|x-x'|\leq\delta \\ x,x'\in\partial B_1(0)}} \left|\Omega^{kl}(x) - \Omega^{kl}(x')\right|}{\delta}\,d\delta < +\infty,$$

required to apply the Calderón–Zygmund inequality as in Theorem 3 of Chapter 2, p.39, in [**Ste70**]. We get

$$\|\mathfrak{w}\|_{L^p(B_R(0))} \leq C\|\widetilde{\mathfrak{f}}\|_{L^p(\mathbb{R}^n)} \leq C\,\|\mathfrak{f}\|_{L^p(B_R(0))} \tag{1.18}$$

for some constant C depending solely on n and p.

Now, for $x \in B_{R/2}(0)$, let

$$\mathfrak{h}(x) = \mathfrak{u}(x) - \mathfrak{w}(x).$$

Taking the Laplacian, we see that

$$\Delta\mathfrak{h} = \Delta\mathfrak{u} - \Delta\mathfrak{w} = \operatorname{div}(\operatorname{div}\widetilde{\mathfrak{f}}) - \operatorname{div}\operatorname{div}\widetilde{\mathfrak{f}} = 0,$$

i.e. $\mathfrak{h}$ is harmonic in $B_{R/2}(0)$. Moreover, $\mathfrak{h}$ solves in $B_{R/2}(0)$

$$D\mathfrak{h} = D\mathfrak{u} - D\mathfrak{w} = \operatorname{div}\widetilde{\mathfrak{f}} - D\mathfrak{w} = \operatorname{div}(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot \mathrm{id}),$$

where id denotes the $n \times n$ identity matrix. Notice that, by our considerations above,

$$\|\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id}\|_{L^p(B_{R/2}(0))} \leq \|\widetilde{\mathfrak{f}}\|_{L^p(B_{R/2}(0))} + n\,\|\mathfrak{w}\|_{L^p(B_{R/2}(0))} \leq C\,\|\mathfrak{f}\|_{L^p(B_R(0))}\,,$$

where C depends only on n and p. Since $\mathfrak{h}$ is harmonic, so is $D\mathfrak{h}$, and we have by the mean value property for all $x \in B_{R/4}(0)$, all $\rho \in (\frac{R}{8}, \frac{R}{4})$ and all i

$$D_i\mathfrak{h}(x) = \fint_{B_\rho(x)} D_i\mathfrak{h} = \fint_{B_\rho(x)} \operatorname{div}(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i = \frac{n}{\rho}\fint_{\partial B_\rho(x)} \sum_{k=1}^{n} \left(\widetilde{\mathfrak{f}}^k{}_i - \mathfrak{w}\delta^k{}_i\right)\left(\nu^{\mathrm{ext}}\right)^k,$$

where the last equality follows from the Gauss Theorem and ν^{ext} denotes the outward unit normal to $\partial B_\rho(x)$. Thus, since $\rho > R/8$, we have for all $x \in B_{R/4}(0)$, all $\rho \in (\frac{R}{8}, \frac{R}{4})$ and all i

$$|D_i\mathfrak{h}| \leq \frac{1}{\widetilde{\omega}_n\rho^n}\int_{\partial B_\rho(x)} \left|(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i\right| \leq \frac{8^n}{\widetilde{\omega}_n R^n}\int_{\partial B_\rho(x)} \left|(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i\right|.$$

Integrating with respect to ρ and using the Hölder inequality we get

$$\begin{aligned}
\frac{R}{8}\,|D_i\mathfrak{h}| &\leq \frac{8^n}{\widetilde{\omega}_n R^n}\int_{\frac{R}{8}}^{\frac{R}{4}}\int_{\partial B_\rho(x)} \left|(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i\right|\,d\rho \\
&= \frac{8^n}{\widetilde{\omega}_n R^n}\left\|(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i\right\|_{L^1(B_{R/4}(x)\setminus B_{R/8}(x))} \\
&\leq \frac{8^n}{\widetilde{\omega}_n R^n}\left\|(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i\right\|_{L^1(B_{R/2}(0))} \\
&\leq \frac{8^n}{\widetilde{\omega}_n R^n}\left(\frac{\widetilde{\omega}_n R^n}{2^n}\right)^{1-1/p}\left\|(\widetilde{\mathfrak{f}} - \mathfrak{w}\cdot\mathrm{id})_i\right\|_{L^p(B_{R/2}(0))}.
\end{aligned}$$

It follows that there is a constant C, depending only on n and p, such that for all $x \in B_{R/4}(0)$

$$|D\mathfrak{h}| \leq CR^{-1-n/p}\,\|\mathfrak{f}\|_{L^p(B_R(0))}\,.$$

Using the Poincaré inequality for balls (see, e.g., [**Eva98**, Thm.2, §5.8.1, p.276]), we obtain

$$\begin{aligned}
\left\|\mathfrak{h} - \fint_{B_{R/4}(0)}\mathfrak{h}\right\|_{L^p(B_{R/4}(0))} &\leq C_1R\,\|D\mathfrak{h}\|_{L^p(B_{R/4}(0))} \\
&\leq C_2R^{-\frac{n}{p}}R^{\frac{n}{p}}\,\|\mathfrak{f}\|_{L^p(B_R(0))} = C_2\,\|\mathfrak{f}\|_{L^p(B_R(0))}\,,
\end{aligned}$$

for some constants C_1 and C_2 depending on n and p. Since $\mathfrak{u} = \mathfrak{w} + \mathfrak{h}$ and with the help of the estimate (1.18) for the potential $\mathfrak{w}$, we arrive at

$$\left\|\mathfrak{u} - \fint_{B_{R/4}(0)}\mathfrak{u}\right\|_{L^p(B_{R/4}(0))} \leq C\,\|\mathfrak{f}\|_{L^p(B_R(0))}\,,$$

for some constant C depending on n and p alone. This proves Proposition 1.11. $\square$

6. Proof of Lemma 1.7

Let $\Sigma \subset \mathbb{R}^{n+1}$ be as in Lemma 1.7 and choose $q \in \Sigma$. We first show some estimates on the size of a geodesic ball $\mathcal{B}^g_\rho(q)$ with radius $\rho \leq r_0$ centred at q. Without loss of generality, we may assume that $q = 0 \in \mathbb{R}^{n+1}$ and that Σ is rotated in such a way that a portion of it is parametrised as the graph of a smooth Lipschitz function $u : D_{r_0}(0) \subset \mathbb{R}^n \to \mathbb{R}$, with $u(0) = 0$ and Lipschitz constant $Lip(u) \leq L$.

Claim 1. *The geodesic sphere $\partial\mathcal{B}^g_\rho(q)$ with radius $\rho \leq r_0$ centred at q is contained in the graph of u over the closed ring $\overline{D_\rho(0)} \setminus D_{\rho/(1+L)}(0)$.*

Proof. Let $\overline{d}_g(x, y)$ denote the geodesic distance from $(x, u(x))$ to $(y, u(y))$. For each y such that $(y, u(y)) \in \partial\mathcal{B}^g_\rho(q)$ we have $\overline{d}_g(0, y) = \rho$. Moreover,

$$\overline{d}_g(0, y) \geq |(y, u(y)) - (0, 0)| = \sqrt{y^2 + u(y)^2} \geq |y|$$

and

$$\overline{d}_g(0, y) \leq \int_0^1 \sqrt{g(\partial_t\gamma, \partial_t\gamma)}\, dt,$$

where γ is any curve $\gamma : [0, 1] \to \Big(D_{r_0}(0), u\big(D_{r_0}(0)\big)\Big)$, $t \mapsto \gamma(t)$, joining $(0, 0)$ and $(y, u(y))$. The second estimate follows from the definition of $\overline{d}_g(0, y)$ as the infimum of the right–hand side taken over all such curves. Choosing $\gamma(t) = (ty, u(ty))$, we obtain with the help of equation (1.13) in the proof of Lemma 1.3 (Section 4) and using that u is Lipschitz (D denoting derivation with respect to the Cartesian coordinates of $\mathbb{R}^n$)

$$\begin{aligned}
\int_0^1 \sqrt{g(\partial_t\gamma, \partial_t\gamma)}\, dt &= \int_0^1 \left(|y|^2 + \left(\sum_{k=1}^n D_k u(ty) y^k\right)^2\right)^{1/2} dt \\
&\leq \int_0^1 \left(|y|^2 + |Du(ty)|^2 |y|^2\right)^{1/2} dt \leq \int_0^1 |y| \sqrt{1 + L^2}\, dt \\
&\leq |y| (1 + L).
\end{aligned}$$

Thus we have

$$|y| \leq \rho \qquad \text{and} \qquad |y| \geq \frac{\rho}{1 + L},$$

proving the claim. $\square$

It follows immediately

Claim 2. *The volume $\mathrm{vol}_n(\mathcal{B}^g_\rho(q))$ of the geodesic ball $\mathcal{B}^g_\rho(q)$ with radius $\rho \leq r_0$ and centre q is bounded by*

$$\frac{\widetilde{\omega}_n \rho^n}{(1 + L)^n} \leq \mathrm{vol}_n(\mathcal{B}^g_\rho(q)) \leq (1 + L)\widetilde{\omega}_n \rho^n,$$

where $\widetilde{\omega}_n = \mathrm{vol}_n(D_1(0))$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Proof. From equation (1.13) in the proof of Lemma 1.3 (Section 4) follows that the volume form σ_g on $\mathcal{B}^g_\rho(q)$ is given by

$$\sigma_g = \sqrt{\det g} = \sqrt{1 + |Du|^2}.$$

With Claim 1, we immediately get by the Lipschitz property of u

$$\mathrm{vol}_n(\mathcal{B}^g_\rho(q)) \geq \int_{D_{\rho/(1+L)}(0)} \sqrt{1 + |Du|^2} \geq \widetilde{\omega}_n \left(\frac{\rho}{1+L}\right)^n$$

and

$$\mathrm{vol}_n(\mathcal{B}^g_\rho(q)) \leq \int_{D_\rho(0)} \sqrt{1 + |Du|^2} \leq \widetilde{\omega}_n \rho^n \sqrt{1 + L^2} \leq (1+L)\widetilde{\omega}_n \rho^n,$$

which proves the Claim. □

An argument similar to the one in the proof of Claim 1 shows

Claim 3. *Let $x \in D_{r_0}(0)$, $q' = (x, u(x))$ and $0 < \rho < r_0 - |x|$. Then the geodesic ball $\mathcal{B}^g_\rho(q')$ is contained in the graph of u over the open ring $D_{|x|+\rho}(0) \setminus \overline{D_{|x|-\rho}(0)}$.*

Proof. For each y such that $(y, u(y)) \in \mathcal{B}^g_\rho(q')$ we have $\overline{d}_g(x, y) < \rho$. Moreover,

$$\overline{d}_g(x, y) \geq |(y, u(y)) - (x, u(x))| = \sqrt{|y - x|^2 + |u(y) - u(x)|^2} \geq |y - x| \geq \big||y| - |x|\big|.$$

Consequently,

$$|x| - \rho < |y| < |x| + \rho,$$

proving the Claim. □

We now proceed with the proof of Lemma 1.7. Let $r_1 = \frac{r_0}{4(1+L)}$ and let

$$\mathfrak{B} = \left\{\mathcal{B}^g_{r_1}(q_j) \;\middle|\; j \in J\right\}$$

be a *maximal* collection of pairwise disjoint geodesic balls in Σ of radius r_1. We will show that the number of balls in this collection is controlled. Moreover, we will show that the collection

$$\widehat{\mathfrak{B}} = \left\{\mathcal{B}^g_{r_0}(q_j) \;\middle|\; j \in J\right\}$$

of geodesic balls of radius r_0 but with same centres q_j has the properties claimed in the lemma to be proved. Clearly, the index set J is finite since Σ is compact. Moreover,

Claim 4. *The cardinality $|J|$ of J is controlled by*

$$|J| \leq \frac{4^n}{\widetilde{\omega}_n} \frac{(1+L)^{2n}}{r_0^n}.$$

Proof. We know from Claim 2 that

$$\mathrm{vol}_n(\mathcal{B}^g_{r_1}(q_j)) \geq \frac{\widetilde{\omega}_n r_1^n}{(1+L)^n} = \frac{\widetilde{\omega}_n r_0^n}{4^n (1+L)^{2n}}.$$

Since $\mathrm{vol}_n(\Sigma) = 1$ and the geodesic balls of radius r_1 are pairwise disjoint, the claim follows. □

Claim 5. *For every $q \in \Sigma$, there exist two distinct indices $j_1, j_2 \in J$, $j_1 \neq j_2$, such that*

$$q \in \mathcal{B}^g_{r_0}(q_{j_1}) \cap \mathcal{B}^g_{r_0}(q_{j_2}).$$

(In particular $\Sigma \subset \cup_{j\in J}\mathcal{B}^g_{r_0}(q_j)$.)

Proof. Let $q \in \Sigma$ be arbitrary. Again, without loss of generality, we may assume that $q = 0 \in \mathbb{R}^{n+1}$ and that a portion of Σ is given as the graph of a smooth L–Lipschitz function $u : D_{r_0}(0) \subset \mathbb{R}^n \to \mathbb{R}$ with $u(0) = 0$.

By Claim 3, there must be a $j_1 \in J$ such that the graph of u over $D_{r_1}(0)$ intersects $\mathcal{B}^g_{r_1}(q_{j_1})$ since, otherwise, by Claim 1, we could add the ball $\mathcal{B}^g_{r_1}(q)$ to $\mathfrak{B}$ to get a bigger collection of pairwise disjoint balls, contradicting the maximality of $\mathfrak{B}$. Notice that, in view of Claim 3, $\mathcal{B}^g_{r_1}(q_{j_1})$ will be contained in the graph of u over $D_{3r_1}(0)$.

Similarly, there must be a $j_2 \in J$, $j_2 \neq j_1$, such that the graph of u over $D_{5r_1}(0)$ intersects $\mathcal{B}^g_{r_1}(q_{j_2})$ since, otherwise, we could fit a ball $\mathcal{B}^g_{r_1}(\widetilde{q})$, for some $\widetilde{q} \in \Sigma$, in the graph of u over the ring $D_{5r_1}(0) \setminus \overline{D_{3r_1}(0)}$. By virtue of Claim 3, that ball would then be disjoint from $\mathcal{B}^g_{r_1}(q_{j_1})$, contradicting again the maximality of $\mathfrak{B}$.

Let y_1 and y_2 be such that $q_{j_1} = (y_1, u(y_1))$ and $q_{j_2} = (y_2, u(y_2))$. Then Claim 3 implies that $|y_1| < 2r_1$ and $|y_2| < 4r_1$.

By Claim 1, then, we have

$$\overline{d}_g(0, y_1) \leq |y_1|\,(1 + L) < \frac{r_0}{2} < r_0$$

and

$$\overline{d}_g(0, y_2) \leq |y_2|\,(1 + L) < r_0.$$

Therefore q is contained in both $\mathcal{B}^g_{r_0}(q_{j_1})$ and $\mathcal{B}^g_{r_0}(q_{j_2})$, finishing the proof of the Claim, and thus the proof of Lemma 1.7. $\square$

CHAPTER 2

The sub–critical and critical cases for convex hypersurfaces

In this chapter we prove our main estimate in the case $p \in (1, n]$ $(n \geq 2)$ for n–dimensional hypersurfaces that are the boundary of some convex domain in $\mathbb{R}^{n+1}$. We also establish the qualitative C^0–closeness to a sphere. The ideas of the proofs are the same as in the super–critical case in Chapter 1. However, we need a different method to "patch together" the local estimates to obtain the global one. It is here then that convexity plays the fundamental role. In fact, we are going to prove a similar result to the one given by Pogorelov in [**Pog73**], obtaining an $(n+1)$–dimensional ring of controlled inner and outer radius that contains the studied hypersurface. This will then enable us to apply Lemma 1.7 to conclude.

Due to the nature of the problem at hand, the restriction we faced in Chapter 1 regarding the necessity to preset a bound on $\|A\|_{L^p(\Sigma)}$ can be avoided.

Contents

1. The main results of this chapter

Our goal is to prove

Theorem 2.1. *Let $n \geq 2$ and $p \in (1, n]$ be given. Then there is a constant $C > 0$, depending only on n and p, such that:*
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed n–dimensional hypersurface with induced Riemannian metric g and such that Σ is the boundary of a convex domain in $\mathbb{R}^{n+1}$, then

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda g\|_{L^p(\Sigma)} \leq C \, \|\mathring{A}\|_{L^p(\Sigma)} \,. \tag{2.1}$$

Since both sides in (2.1) scale identically and none of the assumptions in the theorem are scaling–dependent, we can without loss of generality assume that the n–dimensional volume of Σ be normalised. It is thus sufficient to prove

Theorem 2.1′. *Let $n \geq 2$ and $p \in (1, n]$ be given. Then there is a constant $C > 0$, depending only on n and p, such that:*
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed n–dimensional hypersurface with induced Riemannian metric g and such that

(a) $\mathrm{vol}_n(\Sigma) = 1$

and

(b) *Σ is the boundary of a convex domain in $\mathbb{R}^{n+1}$,*

then

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda g\|_{L^p(\Sigma)} \leq C \, \|\mathring{A}\|_{L^p(\Sigma)} \,. \tag{2.1}$$

Notice that, in view of the following Lemma which we prove at the end of this section, we can choose any constant $c_0 > 0$ and assume without loss of generality that $\|\mathring{A}\|_{L^p(\Sigma)} \leq c_0$ (the author is grateful to C. De Lellis for having brought this to his attention).

Lemma 2.2. *Let $n \geq 2$ and $p \in [1, n]$ be given. Then there exists a constant $C > 0$ depending only on n and p such that:*
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth hypersurface bounding a convex domain and such that $\mathrm{vol}_n(\Sigma) = 1$, then

$$\int_\Sigma |A|^p \leq C \left(1 + \int_\Sigma |\mathring{A}|^p \right) .$$

Indeed, if $\|\mathring{A}\|_{L^p(\Sigma)} > c_0$, we would conclude

$$\begin{aligned} \min_{\lambda \in \mathbb{R}} \|A - \lambda g\|_{L^p(\Sigma)} \leq \|A\|_{L^p(\Sigma)} &\leq C^{\frac{1}{p}} \left(1 + \|\mathring{A}\|^p_{L^p(\Sigma)} \right)^{\frac{1}{p}} \\ &\leq C^{\frac{1}{p}} \left(c_0^{-p} + 1 \right)^{\frac{1}{p}} \|\mathring{A}\|_{L^p(\Sigma)} \,, \end{aligned}$$

and Theorem 2.1′ would be proved. It is therefore enough to prove the weaker

Theorem 2.3. *Let $n \geq 2$, $p \in (1, n]$ and $c_0 > 0$ be given. Then there is a constant $C > 0$, depending only on n, p and c_0, such that:*
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed n–dimensional hypersurface with induced Riemannian metric g and such that

(a) $\mathrm{vol}_n(\Sigma) = 1$,

(b) Σ *is the boundary of a convex domain in* $\mathbb{R}^{n+1}$

and

$$\text{(c)}\quad \|\mathring{A}\|_{L^p(\Sigma)} = \left(\int_\Sigma |\mathring{A}|^p\right)^{\frac{1}{p}} \leq c_0,$$

then

$$\min_{\lambda\in\mathbb{R}} \|A - \lambda g\|_{L^p(\Sigma)} \leq C\,\|\mathring{A}\|_{L^p(\Sigma)}\,.$$

The idea of its proof is exactly the same as the one in the proof of Theorem 1.1 of the previous chapter. However, there is one difference. Then, in order to apply the technical Lemma 1.7 that yields a suitable covering of the hypersurface with geodesic balls, we had to obtain a uniform lower bound on the radii of the balls over which the hypersurface can locally be represented as a Lipschitz graph with given Lipschitz constant. We did this by invoking a Morrey–type estimate (Lemma A.1), which, obviously, does not apply here. It turns out, though, that we can apply Lemma 1.7 directly, thanks to convexity. In fact, as shall be sufficient, we will prove that a hypersurface of the type considered in Theorem 2.3 above is contained in an $(n+1)$–dimensional ring (or spherical shell) where we have (some) control over the inner and the outer radius (namely that they depend only on the data given in the assumptions of the theorem).

We thereby generalise a result given by A. Pogorelov in [**Pog73**, §VII.9, p.493], who proves a theorem stating in a quantitative manner that a convex two–dimensional surface, for which the ratio of the two principal radii of curvature is sufficiently close to one in each point, must be close to a round sphere, in the sense that it lies between two concentric spheres such that the ratio of their radii is also close to one. More precisely, we prove

Proposition 2.4. *Let $n \geq 2$, $p \in (1, n]$ and $c_0 \in (0, +\infty)$ be given. Then there exist $R > r > 0$, depending only on n, p and c_0 such that:*
if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$\mathrm{vol}_n(\partial U) = 1 \qquad \textit{and} \qquad \int_{\partial U} |\mathring{A}|^p \leq c_0,$$

then there exists an $x \in \mathbb{R}^{n+1}$ such that $B_r(x) \subset U \subset B_R(x)$.

In the next section, we quickly demonstrate how this proposition is used to obtain Theorem 2.3, whereas in Section 3, we prove Corollary 2.5 below, which concludes qualitative C^0–closeness to a sphere from the main estimate. The rest of the chapter will then be devoted to proving Proposition 2.4.

Corollary 2.5 (to Theorem 2.1′ and Proposition 2.4)**.** *Let $n \geq 2$, $p \in (1, n]$ and $\epsilon > 0$ be given. Then there is a constant $\delta > 0$, depending only on n, p and ϵ, such that:*
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed n–dimensional hypersurface such that

(a) $\mathrm{vol}_n(\Sigma) = 1$,

(b) Σ *bounds a convex domain in* $\mathbb{R}^{n+1}$

and

(c) $\|\mathring{A}\|_{L^p(\Sigma)} < \delta$,

then

$$d_{\mathrm{HD}}\big(\Sigma, \partial B_{\rho_0}(x)\big) < \epsilon, \qquad \textit{for some } x \in \mathbb{R}^{n+1},$$

where $\rho_0 = (\mathrm{vol}_n(S^n))^{-\frac{1}{n}}$ *and* d_{HD} *denotes the Hausdorff distance in Euclidean* $\mathbb{R}^{n+1}$.

Remark 2.6. *The avid reader might notice while working through this chapter, that the chain of implications may easily be adapted to the convex super–critical case to obtain the analogue of the (weaker) Theorem 2.3. Since we do not need (and do not get) any control on the deviation from one of the ratio of the radii found in Proposition 2.4, we can just use Hölder's inequality for the assumed bound on* $\|\mathring{A}\|^p_{L^p(\Sigma)}$*, to establish that* Σ *is contained in a spherical shell whose radii* $R > r > 0$ *depend only on* n *and* c_0*. Afterwards, we use these radii as in the proof of Theorem 2.3 to cover* Σ *appropriately. By its qualitative nature, Corollary 2.5 then also extends to all exponents* p*. However, we do not see how to obtain the equivalent of the (stronger) Theorem 2.1′ in that situation, nor how the above reasoning would help when seeking quantitative* C^0*–closeness in the convex super–critical case.*

Proof of Lemma 2.2. For all $q \in \Sigma$, let $0 < \lambda_1(q) \le \lambda_2(q) \le \cdots \le \lambda_n(q)$ denote the eigenvalues of the second fundamental form A (i.e. the principal curvatures) in q. We then have, for all $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned}\left(\int_\Sigma |\lambda_i - \lambda_j|^p\right)^{\frac{1}{p}} &\le \left(\int_\Sigma \left|\lambda_i - \frac{1}{n}H\right|^p\right)^{\frac{1}{p}} + \left(\int_\Sigma \left|\lambda_j - \frac{1}{n}H\right|^p\right)^{\frac{1}{p}}\\ &\le 2\left(\int_\Sigma |\mathring{A}|^p\right)^{\frac{1}{p}}.\end{aligned}$$

On the other hand ($1 \le p \le n$),

$$\int_\Sigma \lambda_1^p \le \underbrace{\mathrm{vol}_n(\Sigma)}_{=1}{}^{\frac{n-p}{n}} \left(\int_\Sigma \lambda_1^n\right)^{\frac{p}{n}} \le \left(\int_\Sigma \lambda_1 \cdots \lambda_n\right)^{\frac{p}{n}} = \left(\int_\Sigma \det A\right)^{\frac{p}{n}}.$$

But, since Σ bounds a convex region (cf., e.g., [**Sch93**, eqn.(2.5.29), p.112] or [**CL57**, theorems 3 or 4]),

$$\int_\Sigma \det A = \int_\Sigma \det d\nu = \int_{S^n} 1 = \mathrm{vol}_n(S^n),$$

where we denoted by ν the outer unit normal of Σ (i.e. its Gauss map). We conclude

$$\begin{aligned}\left(\int_\Sigma |A|^p\right)^{\frac{1}{p}} &= \left(\int_\Sigma \left(\lambda_1^2+\cdots+\lambda_n^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq \left(\int_\Sigma (\lambda_1+\cdots+\lambda_n)^p\right)^{\frac{1}{p}} \\ &= \left(\int_\Sigma \Big((\lambda_1-\lambda_1)+\cdots+(\lambda_n-\lambda_1)+n\lambda_1\Big)^p\right)^{\frac{1}{p}} \\ &\leq \sum_{i=2}^n \left(\int_\Sigma (\lambda_i-\lambda_1)^p\right)^{\frac{1}{p}} + n\left(\int_\Sigma \lambda_1^p\right)^{\frac{1}{p}} \\ &\leq 2(n-1)\left(\int_\Sigma |\mathring{A}|^p\right)^{\frac{1}{p}} + n\big(\mathrm{vol}_n(S^n)\big)^{\frac{p}{n}},\end{aligned}$$

from which the desired estimate follows taking, e.g.,

$$C = 2^p \max\left\{2^p(n-1)^p, n^p\big(\mathrm{vol}_n(S^n)\big)^{\frac{p^2}{n}}\right\}.$$

□

2. Proof of Theorem 2.3

Let Σ be as in Theorem 2.3. The only thing we need to check is how Proposition 2.4 implies the assumptions of Lemma 1.7, the rest of the proof being exactly as in Section 2 of Chapter 1. More precisely, we need to ensure that, for each point $q \in \Sigma$, there is a smooth Lipschitz function $u : D_{r_0}(0) \subset \mathbb{R}^n \to \mathbb{R}$ on a ball of radius $r_0 > 0$ and with Lipschitz constant $L > 0$, such that r_0 and L do not depend on the point q under consideration (nor, indeed, on Σ), and such that a portion of Σ containing q can be parametrised as the graph of u.

So fix $q \in \Sigma$, denote by $\Omega \subset \mathbb{R}^{n+1}$ the open convex domain enclosed by Σ and let e_i, $i \in \{1, \ldots, n+1\}$, refer to the standard basis vectors in $\mathbb{R}^{n+1}$. By Proposition 2.4, there are $R > r > 0$ such that $B_r(x) \subset \Omega \subset B_R(x)$ for some $x \in \Omega$. Without loss of generality, we may assume that $x = 0$ and that $q = -|q|\, e_{n+1}$. We then have $r \leq |q| \leq R$. Now denote by $\pi^1_{-|q|} = \left\{y \in \mathbb{R}^{n+1} \;\middle|\; \langle y, e_{n+1}\rangle = -|q|\right\}$ the n–dimensional hyperplane parallel to the span of $\{e_1, \ldots, e_n\}$ and passing through q. Then a portion of Σ containing q can be written as the graph of a smooth convex function u on $D_r = \overline{B_r(q)} \cap \pi^1_{-|q|}$. Moreover, we have $\|u\|_{L^\infty(D_r)} \leq R$, since $B_r(0) \subset \Omega \subset B_R(0)$. It is then easy to see that u is L–Lipschitz on D_ρ, for all $\rho \in \left(0, \frac{r}{2}\right]$, with $L = \frac{4R}{r}$ (see, for example, [**RV74**, Theorem A]). Since q was arbitrary, the assumptions of Lemma 1.7 are met.

As a result, we can cover Σ with N geodesic balls of radius $2r_0 = r/8$, where N depends only on n, p and (through L and r_0) on c_0, and such that the local estimate (1.5) holds for all $r \leq r_0$. Moreover, the elements of this cover will have large enough overlap, in the sense that the intersection of two neighbouring balls of the cover will contain a geodesic ball of radius r_0. This then enables us to argue, once again, that the λs in (1.5) differ at most by $C' \|\mathring{A}\|_{L^p(\Sigma)}$, where C' depends only on n, p and

c_0 (compare with (1.9)). The global estimate then follows. For more details, review the end of the proof laid out in Section 2 of Chapter 1. □

3. Proof of Corollary 2.5

As was the case for the proof of Theorem 2.3 in the last section, we want to follow as closely as possible the argument of the super–critical case (cf. Section 3 of Chapter 1).

3.1. Preliminaries. Assume, by contradiction, that Corollary 2.5 were false. Then we would find a sequence $(U_k)_{k\in\mathbb{N}}$ of open convex subsets of $\mathbb{R}^{n+1}$ with smooth boundaries, satisfying, for each $k \in \mathbb{N}$, $\mathrm{vol}_n(\partial U_k) = 1$ and $\int_{\partial U_k} |\mathring{A}|^p \leq c_0$ for some $c_0 > 0$ independent of k, and such that

$$\lim_{k\to\infty} \|\mathring{A}\|_{L^p(\partial U_k)} = 0.$$

Modulo translating each set U_k, Proposition 2.4 together with Lemma 2.2 implies the existence of $R > r > 0$ such that

$$B_r(0) \subset U_k \subset B_R(0) \qquad (\forall k).$$

Picking a subsequence, if necessary, we can without loss of generality assume that the closures $\overline{U_k}$ converge (in the Hausdorff topology) to a closed convex set $V \subset \mathbb{R}^{n+1}$ (see *Blaschke's selection theorem*, as in, e.g., [**Sch93**, Thm.1.8.6, p.50]). Clearly, we will have

$$B_r(0) \subset V \subset \overline{B_R(0)},$$

so that V is non–degenerate. We shall prove that our assumptions imply that ∂V is a sphere.

Before we begin, we make similar observations as in the proof of Corollary 1.2. Applying Theorem 2.1′ to each ∂U_k, we get, for every $k \in \mathbb{N}$, a $\lambda_k \in \mathbb{R}$ such that

$$\|A - \lambda_k g\|_{L^p(\partial U_k)} \leq C \,\|\mathring{A}\|_{L^p(\partial U_k)}\,,$$

where $C > 0$ depends only on n and p. Then, using Theorem 2.1′ and Lemma 2.2, we obtain (cf. eq. (1.10))

$$|\lambda_k| \leq \frac{\big(C'(1 + c_0^p)\big)^{\frac{1}{p}}}{\sqrt{n}} + \frac{Cc_0}{\sqrt{n}},$$

where $C' > 0$ is the constant from Lemma 2.2 that depends only on n and p. Thus, $(\lambda_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, and, modulo picking a subsequence, we may without loss of generality assume that $\lim_{k\to\infty} \lambda_k = \overline{\lambda} \in \mathbb{R}$.

Also, as we will see in the proof of Proposition 2.4 in the next section, there is a constant $\delta > 0$, depending only on n and p, such that, for each $k \in \mathbb{N}$, $\|A\|_{L^p(\partial U_k)} \geq \delta$ (we apply Corollary 2.8, proved in Section 5, after using Lemma 2.2). With Theorem 2.1′, it then follows that (cf. eq. (1.11))

$$|\lambda_k| \geq \frac{\delta}{\sqrt{n}} - \frac{C}{\sqrt{n}} \,\|\mathring{A}\|_{L^p(\partial U_k)}\,,$$

whence $|\overline{\lambda}| \geq \delta/\sqrt{n} > 0$.

We now start by establishing the convergence to a sphere locally.

3.2. Local convergence. We proceed as in the proof of Corollary 1.2 and construct charts in which portions of ∂U_k are represented by graphs of Lipschitz maps.

Consider any orthonormal system $(x^1, \dots, x^n, x^{n+1})$ at the origin of $\mathbb{R}^{n+1}$, and let $C_- = \left\{x^{n+1} \le 0, (x^1)^2 + \dots + (x^n)^2 \le r^2\right\}$ be the half–infinite cylinder of radius r pointing into the negative x^{n+1}–direction. Since $B_r(0) \subset U_k$, it follows that $\partial U_k \cap C_-$ must be the graph of a function

$$u_k : \overline{D_r(0)} \to \mathbb{R},$$

where $\overline{D_r(0)} = \left\{y \in \mathbb{R}^n \;\middle|\; |y| \le r\right\}$. Obviously, the u_k will be convex, and $U_k \subset B_R(0)$ implies that $\|u_k\|_{L^\infty(\overline{D_r(0)})} \le R$. It is easy to show that this forces the u_k to be $\frac{4R}{r}$–Lipschitz on $\overline{D_{\frac{r}{2}}(0)}$ (see, e.g., [**RV74**, Thm.A]).

We now argue verbatim as in Section 3, replacing the Lipschitz constant there by $\frac{4R}{r}$ and the radius of the ball by $\frac{r}{2}$, to conclude that, locally, a subsequence of $(\partial U_k)_{k\in\mathbb{N}}$ converges to portions of spheres. More precisely, in the present situation (using that we already know that the $\overline{U}_k$ converge) we establish that $\partial V \cap C'_-$, $C'_- = \left\{x^{n+1} \le 0, (x^1)^2 + \dots + (x^n)^2 \le r^2/2\right\}$, is the portion of a sphere of radius $|\overline{\lambda}|^{-1}$. We now use this to obtain the global statement.

3.3. Local to global. If we consider a rotation Φ of $\mathbb{R}^{n+1}$, we can argue in exactly the same way as above to conclude that also $\partial V \cap \Phi(C'_-)$ is the portion of a sphere of radius $|\overline{\lambda}|^{-1}$. Choosing Φ close enough to the identity, we obtain that $\partial V \cap C'_- \cap \Phi(C'_-)$ has large enough overlap to establish that the centres of the spheres containing $\partial V \cap C'_-$ and $\partial V \cap \Phi(C'_-)$ must coincide. We then conclude immediately that V *is* a ball of radius $|\overline{\lambda}|^{-1}$, which contradicts our assumption. Hence the corollary holds. ☐

4. Proof of Propostion 2.4

We will, in fact, prove the slightly weaker (notice the bound on $\int_{\partial U} |A|^p$ replacing the bound on $\int_{\partial U} |\mathring{A}|^p$)

Proposition 2.7. *Let $n \ge 2$, $p \in (1, n]$ and $c_0 \in (0, +\infty)$ be given. Then there exist $R > r > 0$, depending only on n, p and c_0 such that:*
if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$\mathrm{vol}_n(\partial U) = 1 \qquad \textit{and} \qquad \int_{\partial U} |A|^p \le c_0,$$

then there exists an $x \in \mathbb{R}^{n+1}$ such that $B_r(x) \subset U \subset B_R(x)$.

In view of Lemma 2.2 in the first section, this is, indeed, sufficient for obtaining Proposition 2.4. The proof of Proposition 2.7, on the other hand, will be carried out by induction over $n \ge 2$. At the induction step, the following corollary, giving a lower bound on $\int_{\partial U} |A|^p$, will play a crucial role

Corollary 2.8 (to Proposition 2.7)**.** *Let $n \geq 2$ and $p \in (1, n]$ be given. Then there is a constant $\delta > 0$ depending only on n and p such that:*
if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies $\operatorname{vol}_n(\partial U) = 1$, then

$$\int_{\partial U} |A|^p \geq \delta.$$

We prove it in Section 5. Also, we show the two sought–after inclusions of Proposition 2.7 in two separate lemmas, the proofs of which are deferred to sections 6 and 7, respectively.

Lemma 2.9. *Let $n \geq 2$, $p \in (1, n]$ and $c_0 \in (0, +\infty)$ be given. Then there exists a constant $D > 0$, depending only on n, p and c_0, such that:*
if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$\operatorname{vol}_n(\partial U) = 1 \qquad \textit{and} \qquad \int_{\partial U} |A|^p \leq c_0,$$

then $\operatorname{diam} U \leq D$ (where $\operatorname{diam} U$ denotes the diameter of U in $\mathbb{R}^{n+1}$).

Lemma 2.10. *Let $n \geq 2$, $p \in (1, n]$ and $c_0 \in (0, +\infty)$ be given. Then there exists a constant $r > 0$, depending only on n, p and c_0, such that:*
if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$\operatorname{vol}_n(\partial U) = 1 \qquad \textit{and} \qquad \int_{\partial U} |A|^p \leq c_0,$$

then there is an $x \in \mathbb{R}^{n+1}$ such that $B_r(x) \subset U$.

Clearly, both lemmas together imply Proposition 2.7. Also, Lemma 2.10 will be a consequence of Lemma 2.9. The proof of Lemma 2.9 in dimension n, on the other hand, will rely on us having proved Corollary 2.8 (and thus Proposition 2.7) for all dimensions $n' \in \{2, \ldots, n-1\}$, except for when $n = 2$. The reason why we did not split off the induction basis to a separate statement, is that the method used for proving the two–dimensional case is also useful in some n–dimensional cases. For an easier understanding of how our induction argument works, we give the following overview on the chains of implications.

- Induction base ($n = 2$):

$$\underbrace{\text{Lem.2.9}\Big|_2 \implies \text{Lem.2.10}\Big|_2}_{\implies \text{Prop.2.7}\big|_2}$$

- Induction step ($(n-1) \mapsto n$)

$$\left\{\text{Prop.2.7}\Big|_{n'}\right\}_{n' \in \{2,\ldots,n-1\}}$$
$$\implies \left\{\text{Cor.2.8}\Big|_{n'}\right\}_{n' \in \{2,\ldots,n-1\}} \implies \underbrace{\text{Lem.2.9}\Big|_n \implies \text{Lem.2.10}\Big|_n}_{\implies \text{Prop.2.7}\big|_n}$$

∎

5. Proof of Corollary 2.8

The following proof is a variant of the one of Corollary 2.5 (cf. Section 3). Nevertheless, we expose it in full detail to accommodate those readers who eagerly skipped the qualitative C^0–closeness in order to learn how to prove Theorem 2.1′, first.

5.1. Preliminaries. Assume, by contradiction, that the Corollary were not true. Then there must be a sequence $\left(U_j\right)_{j\in\mathbb{N}}$ of open, convex subsets of $\mathbb{R}^{n+1}$ with smooth boundaries, satisfying, for all $j \in \mathbb{N}$, $\mathrm{vol}_n(\partial U_j) = 1$ and $\int_{\partial U_j} |A|^p \le c_0$ for some $c_0 > 0$ independent of j, and such that

$$\lim_{j\to\infty} \int_{\partial U_j} |A|^p = 0.$$

Modulo translating each set U_j, Proposition 2.7 then implies the existence of $R > r > 0$ such that

$$B_r(0) \subset U_j \subset B_R(0) \qquad (\forall j).$$

Picking a subsequence, if necessary, we can then assume that the closures $\overline{U_j}$ converge (in the Hausdorff topology) to a closed convex set $V \subset \mathbb{R}^{n+1}$ (see *Blaschke's selection theorem*, Theorem 1.8.6 on p.50, in [**Sch93**]). Clearly, we will have

$$B_r(0) \subset V \subset \overline{B_R(0)},$$

i.e., V is non–degenerate. We shall prove that our assumptions imply that ∂V is contained in an affine subspace of $\mathbb{R}^{n+1}$, contradicting the above inclusions because of the convexity of V. We first argue locally, showing that, for every $q \in \partial V$, there is a neighbourhood W of q and an affine space $E \subset \mathbb{R}^{n+1}$, such that $\partial V \cap W \subset E$.

5.2. Local convergence. Consider any orthonormal system $\left(x^1, \dots, x^n, x^{n+1}\right)$ at the origin $0 \in \mathbb{R}^{n+1}$, and let $C_- = \left\{x^{n+1} \le 0, (x^1)^2 + \dots + (x^n)^2 \le r^2\right\}$ be the half–infinite cylinder of radius r pointing into the negative x^{n+1}–direction. Since $B_r(0) \subset U_j$, it follows that $\partial U_j \cap C_-$ must be the graph of a function

$$u_j : \overline{D_r(0)} \to \mathbb{R},$$

where $\overline{D_r(0)} = \left\{y \in \mathbb{R}^n \;\middle|\; |y| \le r\right\}$. Obviously, the u_j will be convex, and $U_j \subset B_R(0)$ implies that $\|u_j\|_{L^\infty\left(\overline{D_r(0)}\right)} \le R$. It is easy to show that this forces the u_j to be $\frac{4R}{r}$–Lipschitz on $\overline{D_{\frac{r}{2}}(0)}$ (see, for example, [**RV74**, Theorem A]), and hence, by smoothness, $\|Du_j\|_{L^\infty\left(\overline{D_{r/2}(0)}\right)} \le \frac{4R}{r}$. Now remember that, for the graph of a function φ, the second fundamental form A_φ is given by (see also the proof of Lemma 1.3 in Section 4 of Chapter 1)

$$A_\varphi = \frac{\operatorname{Hess}\varphi}{\sqrt{1 + |D\varphi|^2}},$$

so that

$$|\operatorname{Hess}\varphi| \le |A_\varphi| \sqrt{1 + |D\varphi|^2}.$$

As a consequence, $\|Du_j\|_{L^\infty\left(\overline{D_{r/2}(0)}\right)} \leq \frac{4R}{r}$ and $\lim_{j\to\infty}\int_{\partial U_j}|A|^p = 0$ imply that

$$\|\mathrm{Hess}\, u_j\|_{L^p\left(\overline{D_{r/2}(0)}\right)} \leq \left\|A_{u_j}\right\|_{L^p\left(\overline{D_{r/2}(0)}\right)} \sqrt{1+16\frac{R^2}{r^2}} \overset{j\to\infty}{\longrightarrow} 0.$$

Since *all* u_j are smooth, $\frac{4R}{r}$–Lipschitz and bounded by R on $\overline{D_{r/2}(0)}$, $(u_j)_{j\in\mathbb{N}}$ is a pointwise bounded, equicontinuous sequence in the space of continuous, real–valued functions on the compact domain $\overline{D_{r/2}(0)}$. Then the Ascoli–Arzelà–Theorem (see, e.g., [**Rud91**, Thm.A5, p.394]) implies the existence of a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ that converges uniformly on $\overline{D_{r/2}(0)}$ to a continuous function $\overline{u}$. Since the u_{j_k} are Lipschitz and $u_{j_k} \longrightarrow \overline{u}$ $(k\to\infty)$ uniformly, $\overline{u}$ will be Lipschitz with the same constant $(\frac{4R}{r})$. Now, for any $i\in\{1,\dots,n\}$, we have

$$\|D_i u_{j_k}\|_{L^\infty\left(\overline{D_{r/2}(0)}\right)} \leq \frac{4R}{r} \qquad (\forall k)$$

and

$$\|D(D_i u_{j_k})\|_{L^1\left(\overline{D_{r/2}(0)}\right)} \leq \left(\mathrm{vol}_n\left(\overline{D_{r/2}(0)}\right)\right)^{1-1/p} \|D(D_i u_{j_k})\|_{L^p\left(\overline{D_{r/2}(0)}\right)} \overset{k\to\infty}{\longrightarrow} 0.$$

Consequently, by Rellich–Kondrachov (see, e.g., [**Eva98**, Thm.1, §5.7, p.272]), there is a subsequence $(u_{j_{k_l}})_{l\in\mathbb{N}} \subset (u_{j_k})_{k\in\mathbb{N}}$ and an L^1–function v_i such that $D_i u_{j_{k_l}} \overset{l\to\infty}{\longrightarrow} v_i$ in $L^1\left(\overline{D_{r/2}(0)}\right)$. Moreover, since $D(D_i u_{j_{k_l}}) \overset{l\to\infty}{\longrightarrow} 0$ in L^1, we have that $Dv_i = 0$ in the sense of distributions (see, e.g., [**GT01**, Thm.7.4, p.150]), implying that $v_i = c_i$ almost everywhere for some constant $c_i\in\mathbb{R}$ (see, e.g., [**LL97**, Thm.6.11, p.138]). But since $u_{j_{k_l}} \overset{l\to\infty}{\longrightarrow} \overline{u}$ uniformly, and hence in L^1, as well as $D_i u_{j_{k_l}} \overset{l\to\infty}{\longrightarrow} v_i$ in L^1, we conclude that $D_i\overline{u} = c_i$ in the sense of distributions. As a consequence, $D\left(\overline{u}(x) - \sum_{i=1}^n c_i x^i\right) = 0$ in the sense of distributions, and we conclude that $\overline{u}(x) - \sum_{i=1}^n c_i x^i = b$ almost everywhere for some constant $b\in\mathbb{R}$. By the continuity of $\overline{u}$, we see that $\overline{u}(x) = b + \sum_{i=1}^n c_i x^i$ everywhere, so that $\overline{u}$ is, in fact, an affine function. Writing $C'_- = \left\{x^{n+1}\leq 0, (x^1)^2+\dots+(x^n)^2\leq r^2/4\right\}$, it then follows that $\partial V\cap C'_-$ is contained in an affine hyperplane E. In the next step, we show how to conclude the global statement.

5.3. Local to global. If we consider a rotation Φ of $\mathbb{R}^{n+1}$, we can argue in the same way as above to show that $\partial V\cap\Phi(C'_-)$ is contained in an affine hyperplane F. But if Φ is sufficiently close to the identity, $\partial V\cap C'_-\cap\Phi(C'_-)$ will have positive area, from which we conclude that $E = F$. It is then immediate to see that ∂V is, as a whole, contained in the affine hyperplane E, which is exactly what we claimed.

As already mentioned, this, together with the convexity of V, is incompatible with the inclusions

$$B_r(0)\subset V\subset\overline{B_R(0)},$$

thus finishing the proof of the corollary. □

6. Proof of Lemma 2.9

6.1. Some notation. Before starting the proof, we introduce some notations. Assume $(x^1, \dots, x^{n+1})$ is any orthonormal system in $\mathbb{R}^{n+1}$. For $m \in \{1, \dots, n+1\}$, let $\mathrm{P}^m : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $(x^1, \dots, x^{n+1}) \mapsto (x^{n-m+2}, \dots, x^{n+1})$, denote the projection onto the last m coordinates. For $r \in \mathbb{R}^m$, we define π_r^m as the codimension m hyperplane $\left\{x \in \mathbb{R}^{n+1} \,\middle|\, \mathrm{P}^m(x) = r\right\}$, and ω_0^m as the orthogonal complement $(\pi_0^m)^\perp$ of π_0^m in $\mathbb{R}^{n+1}$. ω_0^m is thus the m–dimensional hyperplane passing through the origin with the first $n-m+1$ coordinates that vanish. For $\theta > 0$, let

$$R^m(\theta) = \left\{ (\theta\lambda_1, \dots, \theta\lambda_m) \,\middle|\, \sum_{i=1}^m |\lambda_i| \le 1 \right\}$$

denote the standard m–dimensional cross–polytope (or hyperrhombus) of length θ in $\mathbb{R}^m$ (it is the convex hull of the $2m$ points given by $\pm\theta$ times the standard basis vectors). We have $\mathrm{vol}_m\big(R^m(\theta)\big) = \frac{2^m}{m!}\theta^m$. Notice that, for all $y \in \mathbb{R}^m \setminus R^m(\theta)$, we have that $|y| > \frac{\theta}{\sqrt{m}}$. Finally, denote by $\mathcal{R}^m(\theta)$ the embedding $\left\{x \in \omega_0^m \,\middle|\, \mathrm{P}^m(x) \in R^m(\theta)\right\}$ of $R^m(\theta)$ into $\mathbb{R}^{n+1}$.

6.2. Preliminaries. Let $n \ge 2$, $p \in (1, n]$ and $c_0 > 0$ be given. If $n \ge 3$, we shall assume that Proposition 2.7 has already been proved for all dimensions $n' \in \{2, \dots, n-1\}$. Since, for all $\tilde{p} \in \big(1, \min\{2, p\}\big]$, we have by Hölder's inequality that

$$\left(\int_{\partial U} |A|^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} \le \left(\int_{\partial U} |A|^p \right)^{\frac{1}{p}},$$

for any open convex set $U \subset \mathbb{R}^{n+1}$ with smooth boundary and $\mathrm{vol}_n(\partial U) = 1$, we may as well assume without loss of generality that $p \in (1, 2]$.

We prove Lemma 2.9 by contradiction. So assume the statement were false. Then there must be a sequence $(U_k)_{k \in \mathbb{N}}$ of open, convex sets with smooth boundary satisfying

(a) $\mathrm{vol}_n(\partial U_k) = 1$,

(b) $\displaystyle\int_{\partial U_k} |A|^p \le c_0$,

(c) $\mathrm{diam}\, U_k \overset{k\to\infty}{\longrightarrow} +\infty$.

For all k, let $d_k = \mathrm{diam}\, U_k$. Modulo translations and rotations in $\mathbb{R}^{n+1}$, we can without loss of generality assume that $\left(0, \dots, 0, \pm\frac{d_k}{2}\right) \in \overline{U_k}$. Let $R_k^1 = R^1\left(\frac{d_k}{2}\right) = \left[-\frac{d_k}{2}, \frac{d_k}{2}\right]$ and $\mathcal{R}_k^1 = \mathcal{R}^1\left(\frac{d_k}{2}\right)$. Thus we assume that, $\forall k \in \mathbb{N}$, $\mathcal{R}_k^1 \subset \overline{U_k}$, since U_k is convex.

If $n - 1 \ge 2$, let

$$\delta_k^1 = \max_{r \in R_k^1} \mathrm{diam}\left(U_k \cap \pi_r^1\right).$$

Then there are two possibilities:

(i) *either* $\displaystyle\limsup_{k\to\infty} \frac{\delta_k^1}{d_k} = 0,$

(ii) *or* $\limsup_{k\to\infty} \frac{\delta_k^1}{d_k} > 0.$

But since $\delta_k^1 \le d_k$, we may in fact assume that, after perhaps picking a subsequence, we are presented with one of the following two alternatives

(i) *either* $\lim_{k\to\infty} \frac{\delta_k^1}{d_k} = 0,$

(ii) *or* there is a constant $\sigma_2 \in (0,1]$ such that $\lim_{k\to\infty} \frac{\delta_k^1}{d_k} = \sigma_2.$

Now let us assume we are in the second case. Then, for k large enough, we have $\frac{\sigma_2}{2} d_k \le \delta_k^1 \le d_k$. Let $r_k \in R_k^1$ be such that $\operatorname{diam}\left(U_k \cap \pi_{r_k}^1\right) = \delta_k^1$ (R_k^1 is compact). Then, modulo rotations (in $\mathbb{R}^{n+1}$ that leave the $(n+1)$st component invariant) and a restriction to the tail of the sequence, we can without loss of generality assume that $\left(0, \ldots, 0, \frac{\sigma_2}{4} d_k, r_k\right) \in \overline{U_k}$, $\forall k \in \mathbb{N}$. Since $\left(0, \ldots, 0, \pm \frac{d_k}{2}\right) \in \overline{U_k}$, the convexity of U_k then implies that $\left(0, \ldots, 0, \frac{\sigma_2}{8} d_k, 0\right) \in \overline{U_k}$ (for $r_k \in R_k^1$ implicates $|r_k| \le \frac{d_k}{2}$), and thus also that $\left(0, \ldots, 0, \frac{\sigma_2}{16} d_k, \pm \frac{d_k}{4}\right) \in \overline{U_k}$ (cf. Figure 2.1). As a result, modulo translations (in $\mathbb{R}^{n+1}$ that leave the last component invariant), we can without loss of generality assume that, $\forall k \in \mathbb{N}$, $\left(0, \ldots, 0, \pm \frac{\sigma_2}{16} d_k, 0\right) \in \overline{U_k}$ and $\left(0, \ldots, 0, 0, \pm \frac{d_k}{4}\right) \in \overline{U_k}$. By convexity, the convex hull of these four points is then also contained in $\overline{U_k}$, and we obtain that there must be a constant $c_2 \in (0,1]$ such that $\mathcal{R}^2(c_2 d_k) \subset \overline{U_k}$ for all k (take, e.g., $c_2 = \min\left\{\frac{\sigma_2}{16}, \frac{1}{4}\right\}$). Let $R_k^2 = R^2(c_2 d_k)$ and $\mathcal{R}_k^2 = \mathcal{R}^2(c_2 d_k)$. We thus assume that $\mathcal{R}_k^2 \subset \overline{U_k}$, $\forall k \in \mathbb{N}$.

If $n - 2 \ge 2$, let

$$\delta_k^2 = \max_{r \in R_k^2} \operatorname{diam}\left(U_k \cap \pi_r^2\right).$$

Then there are, again, two possibilities:

(i) *either* $\limsup_{k\to\infty} \frac{\delta_k^2}{d_k} = 0,$

(ii) *or* $\limsup_{k\to\infty} \frac{\delta_k^2}{d_k} > 0.$

In the second case, we can argue in an analogous manner to obtain that, without loss of generality, we may assume the existence of a constant $c_3 \in (0,1]$ such that $\mathcal{R}_k^3 = \mathcal{R}^3(c_3 d_k) \subset \overline{U_k}$, $\forall k \in \mathbb{N}$.

Continuing this argument inductively, it is easy to see that, after an appropriate application of translations and rotations, as well as after picking a convenient subsequence, we may without loss of generality assume that

> *there exists an $m \in \{1, \ldots, n-1\}$ and a constant $c_m \in (0,1]$ such that, for all $k \in \mathbb{N}$,*
>
> $$\mathcal{R}^m(c_m d_k) \subset \overline{U_k};$$
>
> *moreover, if $m \le n-2$, then*
>
> $$\limsup_{k\to\infty} \frac{\max_{r \in R^m(c_m d_k)} \operatorname{diam}\left(U_k \cap \pi_r^m\right)}{d_k} = 0.$$

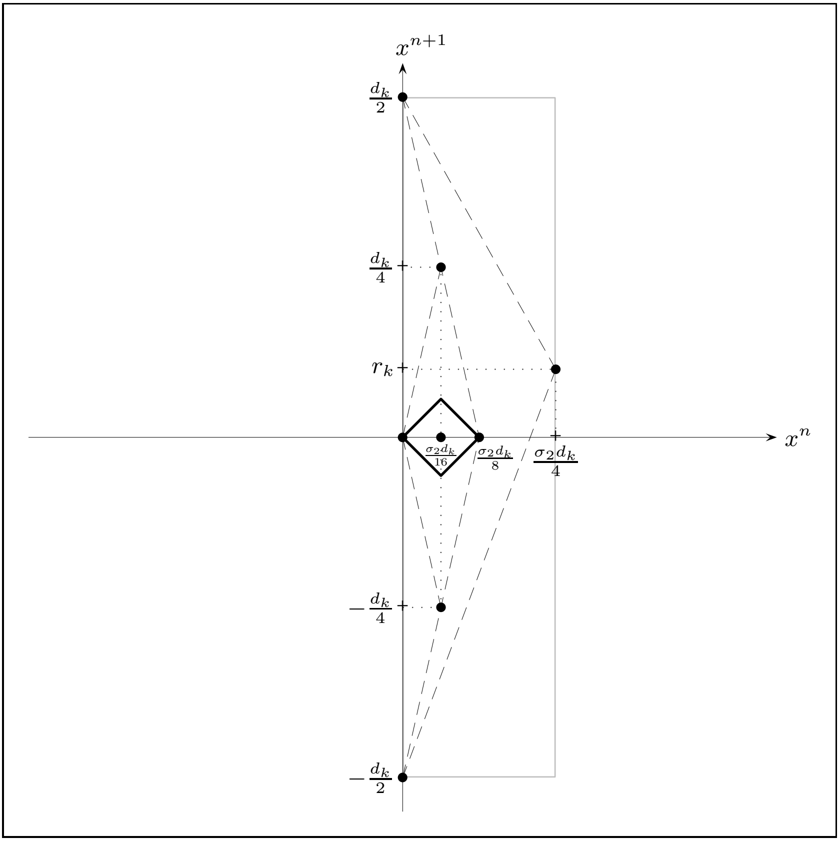

FIGURE 2.1.

As we shall see shortly, even if $m = n - 1$, the second statement still holds true.

6.3. The actual proof. Let $R_k^m = R^m(c_m d_k)$ and $\mathcal{R}_k^m = \mathcal{R}^m(c_m d_k)$.

Claim 1. *For all $k \in \mathbb{N}$, we have*

$$\max_{r \in R_k^m} \operatorname{vol}_{n-m}\left(\partial U_k \cap \pi_r^m\right) \leq \frac{2(m!)}{(c_m)^m (d_k)^m}.$$

PROOF. For each $k \in \mathbb{N}$, let

$$\mu_k = \max_{r \in R_k^m} \operatorname{vol}_{n-m}\left(\partial U_k \cap \pi_r^m\right)$$

and ρ_k be the point in R_k^m where this maximum is attained (R_k^m is compact). Let $\widetilde{R}_k^m$ denote the translation of $R^m\left(\frac{c_m}{2} d_k\right)$ into the point $\frac{\rho_k}{2}$, and let $\widetilde{\mathcal{R}}_k^m =$

$\left\{x \in \omega_0^m \;\middle|\; \mathrm{P}^m(x) \in \widetilde{R}_k^m\right\}$. Notice that $\widetilde{R}_k^m \subset R_k^m$ and $\widetilde{\mathcal{R}}_k^m \subset \mathcal{R}_k^m \subset \overline{U_k}$. Observe also that, for any $y \in \widetilde{\mathcal{R}}_k^m$, we have $y_k + 2(y - y_k) \in \mathcal{R}_k^m$, where $y_k \in \widetilde{\mathcal{R}}_k^m$ is such that $\mathrm{P}^m(y_k) = \rho_k$ (cf. Figure 2.2). Now, for any $y \in \widetilde{\mathcal{R}}_k^m \setminus \{y_k\}$, denote by $C_k(y)$ the cone with base $\partial U_k \cap \pi^m_{\rho_k}$ and tip $2y - y_k$. Then, by convexity, $C_k(y) \subset \overline{U_k}$. Moreover, since $U_k \cap \pi^m_{\rho_k}$ is convex, $C_k(y) \cap \pi^m_{\mathrm{P}^m(y)}$ bounds a convex region and we must have

$$\mathrm{vol}_{n-m}\left(C_k(y) \cap \pi^m_{\mathrm{P}^m(y)}\right) \leq \mathrm{vol}_{n-m}\left(\partial U_k \cap \pi^m_{\mathrm{P}^m(y)}\right).$$

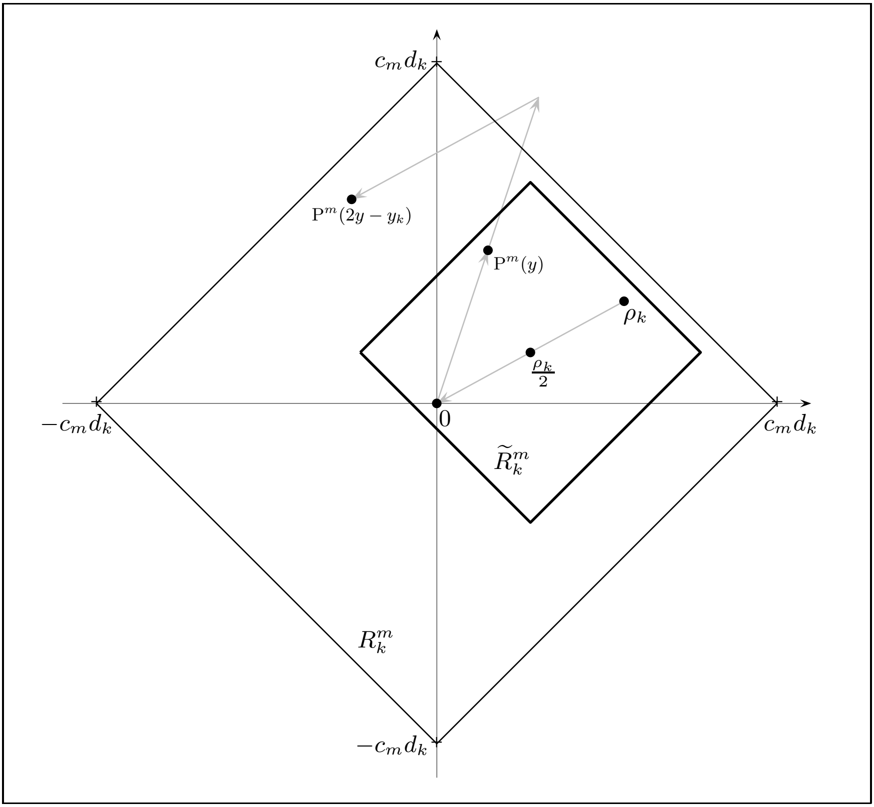

FIGURE 2.2.

Remark 2.11. *The fact that* $\mathrm{vol}(\partial M) \leq \mathrm{vol}(\partial N)$, *whenever* M *is a convex subset of the open set* N *(and assuming both have smooth boundary) follows from the definition of the Hausdorff measure and the fact that the nearest point projection onto* M *is norm–non–increasing — see, e.g., [**BH99**, Prop.2.4(4), p.177], and also [**Cha06**, Ex.III.12(i), p.161].*

On the other hand, we have by construction

$$\mathrm{vol}_{n-m}\left(C_k(y)\cap\pi^m_{\mathrm{P}^m(y)}\right)=\frac{1}{2}\mathrm{vol}_{n-m}\left(\partial U_k\cap\pi^m_{\rho_k}\right)=\frac{\mu_k}{2}.$$

As a consequence, we get for all $y\in\widetilde{\mathcal{R}}^m_k$:

$$\mathrm{vol}_{n-m}\left(\partial U_k\cap\pi^m_{\mathrm{P}^m(y)}\right)\geq\frac{\mu_k}{2}.$$

But then the coarea formula (see, e.g., [**Cha06**, §III.8] or [**Fed69**, §3.2]) yields

$$\begin{aligned}1=\mathrm{vol}_n(\partial U_k)&\geq\mathrm{vol}_n\left(\left\{y\in\partial U_k\;\middle|\;\mathrm{P}^m(y)\in\widetilde{\mathcal{R}}^m_k\right\}\right)\\&=\int_{r\in\widetilde{R}^m_k}\mathrm{vol}_{n-m}\left(\partial U_k\cap\pi^m_r\right)\,dr\geq\frac{\mu_k}{2}\mathrm{vol}_m\left(\widetilde{R}^m_k\right).\end{aligned}$$

And since

$$\mathrm{vol}_m\left(\widetilde{R}^m_k\right)=\mathrm{vol}_m\left(R^m\left(\frac{c_m}{2}d_k\right)\right)=\frac{2^m}{m!}\left(\frac{c_m}{2}d_k\right)^m=\frac{(c_m)^m(d_k)^m}{m!},$$

the claim follows immediately. □

Define

$$\delta_k=\max_{r\in R^m(c_md_k)}\mathrm{diam}\left(U_k\cap\pi^m_r\right),$$

and let $r_k\in R^m_k=R^m(c_md_k)$ be such that $\mathrm{diam}\left(U_k\cap\pi^m_{r_k}\right)=\delta_k$ (R^m_k is compact). If $m\leq n-2$, we have $\lim_{k\to\infty}\frac{\delta_k}{d_k}=0$ by assumption. If, however, $m=n-1$ (as is necessarily the case when $n=2$), then Claim 1 yields

$$\lim_{k\to\infty}\mathrm{vol}_1\left(\partial U_k\cap\pi^{n-1}_r\right)=0,\qquad\forall r\in R^{n-1}_k.$$

But, for each $r\in R^{n-1}_k$, $\partial U_k\cap\pi^{n-1}_r$ is a simple closed C^∞–curve in the two–dimensional hyperplane π^{n-1}_r, whence

$$\mathrm{diam}\left(U_k\cap\pi^{n-1}_r\right)\leq\frac{1}{2}\mathrm{vol}_1\left(\partial U_k\cap\pi^{n-1}_r\right).$$

It follows that $\lim_{k\to\infty}\frac{\delta_k}{d_k}=0$ also in the case $m=n-1$.

Now let

$$\eta_k=\sqrt{m}\delta_k.$$

Since $\lim_{k\to\infty}\frac{\eta_k}{d_k}=0$, we may, modulo picking a subsequence that contains only the tail, without loss of generality assume that $\eta_k<\frac{c_m}{2}d_k$, $\forall k\in\mathbb{N}$. Define

$$\widehat{R}^m_k=R^m\left(c_md_k-\eta_k\right)\quad\text{and}\quad\widehat{\mathcal{R}}^m_k=\mathcal{R}^m\left(c_md_k-\eta_k\right).$$

Then we have

Claim 2. *For all $k\in\mathbb{N}$, $r\in\widehat{R}^m_k$ and $q\in\partial U_k\cap\pi^m_r$, the angle $\angle\left(\nu(q),\pi^m_0\right)$ between the outer unit normal $\nu(q)$ to ∂U_k in q and the hyperplane π^m_0 is less than or equal to $\frac{\pi}{4}$.*

Proof. Fix $r \in \widehat{R}_k^m$ and $q \in \partial U_k \cap \pi_r^m$, and let $y \in \widehat{\mathcal{R}}_k^m$ be such that $\mathrm{P}^m(y) = r$. The outer unit normal $\nu(q)$ to ∂U_k in q then decomposes as $\nu(q) = \nu' + \nu''$, where $\nu' \in \pi_0^m$ and $\nu'' \in (\pi_0^m)^\perp = \omega_0^m$. Clearly,

$$\left|\nu'\right| = \cos\Big(\angle\left(\nu(q), \pi_0^m\right)\Big)$$

and

$$\left|\nu''\right| = \sin\Big(\angle\left(\nu(q), \pi_0^m\right)\Big) = \sqrt{1 - |\nu'|^2}.$$

If $\nu'' = 0$, then there is nothing to prove, so assume $\nu'' \neq 0$. Let $y^* \in \mathcal{R}_k^m$ be such that $r^* = \mathrm{P}^m(y^*) \in \partial R_k^m$ and $\frac{y^*-y}{|y^*-y|} = \frac{\nu''}{|\nu''|}$ (this is possible, since R_k^m contains a ball of radius $\frac{\eta_k}{\sqrt{m}}$ around every $\widetilde{r} \in \widehat{R}_k^m$) — cf. Figure 2.3. Notice that $|y^* - y| \geq \frac{\eta_k}{\sqrt{m}}$, since $y \in \widehat{\mathcal{R}}_k^m$.

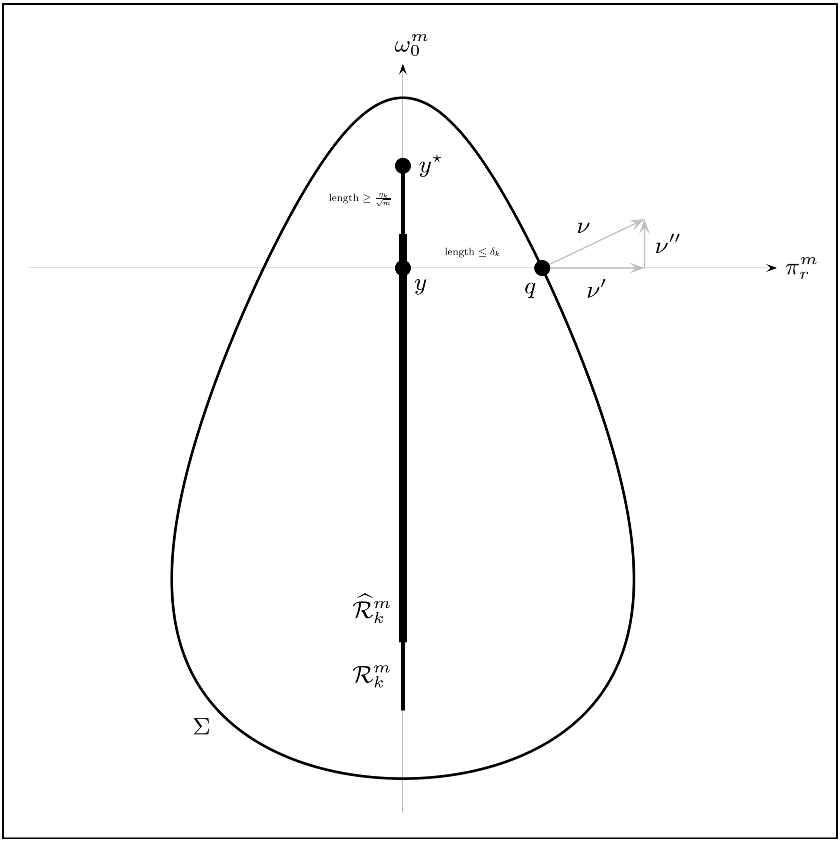

Figure 2.3.

Now, since U_k is convex, we have $\overline{U_k} \subset \left\{z \in \mathbb{R}^{n+1} \;\middle|\; \langle z-q, \nu(q)\rangle \leq 0\right\}$, where, once more, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. Hence,

$$0 \geq \langle y^* - q, \nu(q)\rangle = \langle y^* - y, \nu(q)\rangle + \langle y - q, \nu(q)\rangle,$$

since $y^* \in \overline{U_k}$. But, by construction, we have

$$\langle y^* - y, \nu(q)\rangle = \frac{|y^* - y|}{|\nu''|} \left\langle \nu'', \nu'' \right\rangle = |y^* - y| \sqrt{1 - |\nu'|^2}.$$

Moreover, noticing that $q - y \in \pi_0^m$, we have (observe that $\langle q - y, \nu(q)\rangle \geq 0$, since $y \in \overline{U_k}$)

$$\langle q - y, \nu(q)\rangle = \left\langle q - y, \nu' \right\rangle \leq |q - y| \left|\nu'\right|.$$

We therefore get

$$\operatorname{cotan}\Big(\angle\left(\nu(q), \pi_0^m\right)\Big) = \frac{|\nu'|}{\sqrt{1 - |\nu'|^2}} \geq \frac{|y^* - y|}{|q - y|} \geq \frac{\eta_k}{\sqrt{m}\,|q - y|} = \frac{\delta_k}{|q - y|}.$$

But given that $q, y \in \overline{U_k} \cap \pi_r^m$ and $\operatorname{diam}(U_k \cap \pi_r^m) \leq \delta_k$, we obtain the desired inequality $\operatorname{cotan}\Big(\angle\left(\nu(q), \pi_0^m\right)\Big) \geq 1$. $\square$

We now wish to prove:

Claim 3.

$$\liminf_{k\to\infty} \operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \in \widehat{\mathcal{R}}_k^m\right\}\right) > 0.$$

Proof. Assume first that

$$\lim_{k\to\infty} \operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \notin \widehat{\mathcal{R}}_k^m\right\}\right) = 0.$$

Then there is nothing to prove, since, in that case,

$$\begin{aligned}\liminf_{k\to\infty} &\operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \in \widehat{\mathcal{R}}_k^m\right\}\right) \\ &= \liminf_{k\to\infty} \underbrace{\operatorname{vol}_n(\partial U_k)}_{=1} - \limsup_{k\to\infty} \operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \notin \widehat{\mathcal{R}}_k^m\right\}\right) = 1.\end{aligned}$$

So suppose that $\limsup_{k\to\infty} \operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \notin \widehat{\mathcal{R}}_k^m\right\}\right) > 0$. Modulo taking a subsequence, we may without loss of generality assume that there is a $v_c \in (0, 1)$ such that, for every $k \in \mathbb{N}$, we have

$$\operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \notin \widehat{\mathcal{R}}_k^m\right\}\right) \geq v_c.$$

From the coarea formula, we then have

$$\begin{aligned}v_c &\leq \operatorname{vol}_n \left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \notin \widehat{\mathcal{R}}_k^m\right\}\right) = \int_{r \notin \widehat{R}_k^m} \operatorname{vol}_{n-m}\left(\partial U_k \cap \pi_r^m\right) dr \\ &\leq \sum_{i=n-m+2}^{n+1} \int_{|\rho| \geq \frac{c_m d_k - \eta_k}{\sqrt{m}}} \operatorname{vol}_{n-1}\Big(\partial U_k \cap \left\{y = (y^1, \ldots, y^{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; y^i = \rho\right\}\Big) d\rho,\end{aligned}$$

since $R^m(\theta)$ contains the m–cube of length $\frac{\theta}{\sqrt{m}}$. Notice that all the integrals are well–defined, since the integrands vanish for arguments of integration with length greater than d_k. Since the right–hand side of the above inequality is invariant under renumbering of the coordinates, we may, modulo reflections, without loss of generality assume that

$$\int_{-d_k}^{-\frac{c_m d_k - \eta_k}{\sqrt{m}}} \mathrm{vol}_{n-1}(\partial U_k \cap \pi^1_\rho)\, d\rho \geq \frac{v_c}{2m}.$$

Consequently, for each $k \in \mathbb{N}$, there must be a $\rho_k \in \left[-d_k, -\frac{c_m d_k - \eta_k}{\sqrt{m}}\right]$ such that

$$\mathrm{vol}_{n-1}(\partial U_k \cap \pi^1_{\rho_k}) \geq \frac{v_c}{2\sqrt{m}\big((\sqrt{m}-c_m)d_k + \eta_k\big)} \geq \frac{v_c}{\sqrt{m}(2\sqrt{m}-c_m)d_k}, \tag{2.2}$$

since we had assumed that $\eta_k \leq \frac{c_m}{2} d_k$.

Now remember that, by assumption, $(0,\ldots,0,c_m d_k) \in \overline{U_k}$, since $\mathcal{R}^m_k \subset \overline{U_k}$. Thus, by the convexity of U_k, the whole cone C_k with base $\partial U_k \cap \pi^1_{\rho_k}$ and tip $(0,\ldots,0,c_m d_k)$ must be contained in $\overline{U_k}$. Moreover, since $\partial U_k \cap \pi^1_{\rho_k}$ is convex, we must have

$$\mathrm{vol}_{n-1}(\partial U_k \cap \pi^1_\rho) \geq \mathrm{vol}_{n-1}(C_k \cap \pi^1_\rho) = \frac{c_m d_k - \rho}{c_m d_k - \rho_k} \mathrm{vol}_{n-1}(\partial U_k \cap \pi^1_{\rho_k}) \tag{2.3}$$

for all $\rho \in (\rho_k, c_m d_k]$. But then the coarea formula yields (since $\mathcal{R}^1(c_m d_k - \eta_k) \subset \widehat{\mathcal{R}}^m_k$)

$$\begin{aligned}
&\mathrm{vol}_n\left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \in \widehat{\mathcal{R}}^m_k\right\}\right)\\
&\quad\geq \mathrm{vol}_n\left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \in \mathcal{R}^1(c_m d_k - \eta_k)\right\}\right)\\
&\quad\geq \mathrm{vol}_n\left(\partial U_k \cap \left\{y = (y^1,\ldots,y^{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; y^{n+1} \in [0, c_m d_k - \eta_k]\right\}\right)\\
&\quad= \int_0^{c_m d_k - \eta_k} \mathrm{vol}_{n-1}(\partial U_k \cap \pi^1_\rho)\, d\rho\\
&\quad\overset{(2.2)\&(2.3)}{\geq} \int_0^{c_m d_k - \eta_k} \frac{c_m d_k - \rho}{c_m d_k - \rho_k} \cdot \frac{v_c}{\sqrt{m}(2\sqrt{m}-c_m)d_k}\, d\rho\\
&\quad= \frac{v_c}{2\sqrt{m}(2\sqrt{m}-c_m)(c_m d_k - \rho_k)d_k}\left((c_m d_k)^2 - (\eta_k)^2\right)\\
&\quad\geq \frac{v_c (c_m)^2}{2\sqrt{m}(2\sqrt{m}-c_m)(1+c_m)} - \frac{v_c}{2\sqrt{m}(2\sqrt{m}-c_m)(1+c_m)}\left(\frac{\eta_k}{d_k}\right)^2,
\end{aligned}$$

where the last line follows from $\rho_k \geq -d_k$. Remembering that $\limsup_{k\to\infty} \frac{\eta_k}{d_k} = 0$, we see that the claim holds. □

Henceforth we shall, modulo picking a subsequence, without loss of generality assume that $v_0 \in (0,1)$ is such that

$$\mathrm{vol}_n\left(\partial U_k \cap \left\{y \in \mathbb{R}^{n+1} \;\middle|\; y \in \widehat{\mathcal{R}}^m_k\right\}\right) \geq v_0 \qquad \forall k \in \mathbb{N}.$$

The following final claim then yields a contradiction to the assumption

$$\int_{\partial U_k} |A|^p \leq c_0.$$

Claim 4.

$$\liminf_{k\to\infty} \int_{\partial U_k} |A|^p = +\infty.$$

Proof. Fix $k \in \mathbb{N}$. For all $r \in \widehat{R}_k^m$, let $\Gamma_{k,r}^{n-m} = \partial U_k \cap \pi_r^m$. In the following, denote by $\overline{A}$ the second fundamental form of $\Gamma_{k,r}^{n-m}$ in π_r^m, and by $\overline{\nu}$ the Gauss map of $\Gamma_{k,r}^{n-m}$ in π_r^m. We have to distinguish the two cases: $m = n-1$ and $m \leq n-2$.

If $m = n-1$ (which is necessarily the case when $n = 2$), $\Gamma_{k,r}^1$ is a simple closed C^∞–curve in π_r^{n-1}. Consequently, there are two points q_1 and q_2 in $\Gamma_{k,r}^1$ such that $\overline{\nu}(q_1) = -\overline{\nu}(q_2)$. Corollary A.4, proved in the appendix, then gives

$$\int_\gamma |\overline{A}| \geq |\overline{\nu}(q_1) - \overline{\nu}(q_2)| = 2,$$

for each of the two arcs $\gamma \subset \Gamma_{k,r}^1$ joining q_1 and q_2. As a consequence,

$$\int_{\Gamma_{k,r}^1} |\overline{A}| \geq 2.$$

Using Lemma A.2 of the appendix, together with Claim 2, then yields

$$\int_{\Gamma_{k,r}^1} |A| \geq \frac{1}{\sqrt{2}} \int_{\Gamma_{k,r}^1} |\overline{A}| \geq \sqrt{2}.$$

It thus follows from the coarea formula that

$$\begin{aligned}\int_{\partial U_k} |A| \geq \int_{\partial U_k \cap \left\{ y\in\mathbb{R}^{n+1} \middle| y \in \widehat{\mathcal{R}}_k^{n-1}\right\}} |A| &= \int_{\widehat{R}_k^{n-1}} \left(\int_{\Gamma_{k,r}^1} |A| \right) dr \\ &\geq \sqrt{2}\mathrm{vol}_{n-1}\left(\widehat{R}_k^{n-1}\right).\end{aligned}$$

Since $k \in \mathbb{N}$ was arbitrary and

$$\mathrm{vol}_{n-1}\left(\widehat{R}_k^{n-1}\right) = \mathrm{vol}_{n-1}\left(R^{n-1}(c_{n-1}d_k - \eta_k)\right) \quad \overset{k\to\infty}{\longrightarrow} \quad +\infty,$$

we conclude that, indeed,

$$\liminf_{k\to\infty} \int_{\partial U_k} |A|^p \geq \liminf_{k\to\infty} \left(\Big(\underbrace{\mathrm{vol}_n(\partial U_k)}_{=1}\Big)^{1-p} \left(\int_{\partial U_k} |A| \right)^p \right) = +\infty,$$

if $m = n-1$.

Now assume that $m \le n-2$, and denote by $\widetilde{\Gamma}^{n-m}_{k,r}$ the rescaling of $\Gamma^{n-m}_{k,r}$ such that $\mathrm{vol}_{n-m}\left(\widetilde{\Gamma}^{n-m}_{k,r}\right) = 1$. Also, let $\widetilde{A}$ be the second fundamental form of $\widetilde{\Gamma}^{n-m}_{k,r}$ in π^m_r. Clearly, we have

$$|\widetilde{A}| = \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right)\right)^{\frac{1}{n-m}} |\overline{A}|.$$

As a consequence, using Claim 2 with Lemma A.2, we obtain

$$\begin{aligned}\int_{\widetilde{\Gamma}^{n-m}_{k,r}} |\widetilde{A}|^p &= \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right)\right)^{\frac{p}{n-m}-1} \int_{\Gamma^{n-m}_{k,r}} |\overline{A}|^p \\ &\le \left(\sqrt{2}\right)^p \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right)\right)^{\frac{p}{n-m}-1} \int_{\Gamma^{n-m}_{k,r}} |A|^p .\end{aligned}$$

Remember that we had assumed $p \in (1,2] \subset (1, n-m]$, and that Proposition 2.7 is already proved for every $n' \in \{2, \dots, n-1\}$. We may thus apply Corollary 2.8 to $\widetilde{\Gamma}^{n-m}_{k,r}$, yielding

$$\begin{aligned}\int_{\Gamma^{n-m}_{k,r}} |A|^p &\ge 2^{-\frac{p}{2}} \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right)\right)^{1-\frac{p}{n-m}} \int_{\widetilde{\Gamma}^{n-m}_{k,r}} |\widetilde{A}|^p \\ &\ge 2^{-\frac{p}{2}} \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right)\right)^{1-\frac{p}{n-m}} \delta,\end{aligned}$$

for some $\delta > 0$ depending only on $(n-m) \in \{2, \dots, n-1\}$ and p. The coarea formula then yields

$$\begin{aligned}\int_{\partial U_k} |A|^p &\ge \int_{\partial U_k \cap \left\{ y \in \mathbb{R}^{n+1} \,\middle|\, y \in \widehat{\mathcal{R}}^m_k \right\}} |A|^p = \int_{r \in \widehat{R}^m_k} \left(\int_{\Gamma^{n-m}_{k,r}} |A|^p \right) dr \\ &\ge 2^{-\frac{p}{2}} \delta \int_{r \in \widehat{R}^m_k} \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right) \left(\mathrm{vol}_{n-m}\left(\Gamma^{n-m}_{k,r}\right)\right)^{-\frac{p}{n-m}} \right) dr \\ &\overset{\text{Claim 1}}{\ge} 2^{-\frac{p}{2}} \delta \left(\frac{2(m!)}{(c_m)^m (d_k)^m} \right)^{-\frac{p}{n-m}} \underbrace{\mathrm{vol}_n\left(\partial U_k \cap \left\{ y \in \mathbb{R}^{n+1} \,\middle|\, y \in \widehat{\mathcal{R}}^m_k \right\} \right)}_{\substack{\text{Claim 3} \\ \ge v_0 > 0}} \\ &\ge 2^{-p\left(\frac{1}{2}+\frac{1}{n-m}\right)} \left(\frac{(c_m)^m}{m!} \right)^{\frac{p}{n-m}} \delta v_0 (d_k)^{\frac{pm}{n-m}} \qquad \overset{k\to\infty}{\longrightarrow} \quad +\infty,\end{aligned}$$

from which

$$\liminf_{k\to\infty} \int_{\partial U_k} |A|^p = +\infty.$$

This proves the claim also in the case $m \le n-2$, and Lemma 2.9 is shown. $\square$

7. Proof of Lemma 2.10

Let $n \geq 2$, $p \in (1, n]$ and $c_0 > 0$ be given, and assume the Lemma were false. Then there exists a sequence $(U_k)_{k \in \mathbb{N}}$ of open, convex subsets of $\mathbb{R}^{n+1}$ having smooth boundary, containing the origin, satisfying, for all $k \in \mathbb{N}$,

(a) $\mathrm{vol}_n(\partial U_k) = 1$,

(b) $\displaystyle\int_{\partial U_k} |A|^p \leq c_0$,

(c) $\overline{U_k} \subset \overline{B_R(0)}$ (for some $R > 0$, depending only on n, p and c_0, by virtue of Lemma 2.9),

and with the property that

(d) the $\overline{U_k}$ converge, in the sense of Hausdorff, to a compact convex set V contained in an n–dimensional hyperplane in $\mathbb{R}^{n+1}$.

(As in the proof of Corollary 2.8, the fact that we can assume that the $\overline{U_k}$ — or, at least, a subsequence thereof — converge is a consequence of *Blaschke's selection theorem* ([**Sch93**, Thm.1.8.6, p.50]), whereas the fact that the limit must be contained in a hyperplane follows from our contradiction assumption, namely that

$$\lim_{k \to \infty} \sup_{x \in U_k} \sup \left\{ \rho > 0 \;\middle|\; B_\rho(x) \subset U_k \right\} = 0.)$$

Claim 1. $\dim(V) = n$

Proof. Assume, by contradiction, that $\dim(V) \leq n-1$, and consider, for $\epsilon > 0$, the tubular neighbourhood $V_\epsilon = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \mathrm{dist}(x, V) < \epsilon \right\}$ of V. V_ϵ is an open, convex subset of $\mathbb{R}^{n+1}$ which, by convergence, contains $\overline{U_k}$ for k large enough. On the other hand, since we assumed that $\dim(V) \leq n - 1$, we must have

$$\lim_{\epsilon \searrow 0} \mathrm{vol}_n(\partial V_\epsilon) = 0,$$

for V is bounded. Hence, choosing ϵ small enough, we can assume that $\mathrm{vol}_n(\partial V_\epsilon) \leq \frac{1}{2}$. But given that $\overline{U_k} \subset V_\epsilon$, the convexity of U_k implies

$$\mathrm{vol}_n(\partial U_k) \leq \mathrm{vol}_n(\partial V_\epsilon) \leq \frac{1}{2}, \quad \text{whenever } \epsilon \text{ is small enough and } k \text{ large enough,}$$

(cf. Remark 2.11 on p. 32). This, however, contradicts $\mathrm{vol}_n(\partial U_k) = 1$ $(\forall k \in \mathbb{N})$, and the claim is proved. ☐

Without loss of generality, we may assume that $V \subset \left\{ (z, 0) \in \mathbb{R}^{n+1} \;\middle|\; z \in \mathbb{R}^n \right\} = \pi_0^1$. We define

$$I = \left\{ x \in \pi_0^1 \;\middle|\; B_\eta(x) \cap \pi_0^1 \subset V \text{ for some } \eta > 0 \right\},$$

the "interior", and

$$B = V \setminus I,$$

the "boundary" of V in π_0^1. For $x = (z, 0) \in \pi_0^1$ $(z \in \mathbb{R}^n)$, let

$$l_x = \left\{ (z, \rho) \;\middle|\; \rho \in \mathbb{R} \right\}$$

denote the "vertical" line passing through x. Then we have

Claim 2. *For any compact $K \subset I$, any $x = (z, 0) \in K$ and any $k \in \mathbb{N}$ large enough, $l_x \cap \overline{U_k}$ is a closed, non–degenerate segment joining the points $\big(z, a(z)\big)$ and $\big(z, b(z)\big)$, where $a(z) < b(z)$.*

PROOF. Fix a compact set $K \subset I$. Clearly, by convexity, $l_x \cap \partial U_k$ will never consist of more than two points, regardless of the $x \in \pi_0^1$ we choose. But assume that, for each k, there were an $x_k = (z_k, 0) \in K$ such that $l_{x_k} \cap \partial U_k$ consists of at most one point. Then

(i) *either* $l_{x_k} \cap \overline{U_k} = \emptyset$,
(ii) *or* l_{x_k} is tangent to ∂U_k.

Consider, for each $k \in \mathbb{N}$, the projection $V_k = \mathrm{P}^m(U_k)$ of U_k onto π_0^1. Then V_k is a convex subset of π_0^1 which is relatively open in π_0^1, and, in both of the cases above, $x_k = (z_k, 0) \notin V_k$ (but, possibly, $x_k \in \overline{V_k}$). Then the theorem of Hahn–Banach (see, e.g., Theorem I.6 on p.5 in [**Bre83**]) ensures the existence of a unit vector $e_k \in S^{n-1}$ such that

$$V_k \subset \left\{ (w, 0) \in \mathbb{R}^{n+1} \;\middle|\; \langle (w - z_k), e_k \rangle_{\mathbb{R}^n} \leq 0 \right\},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the standard scalar product in $\mathbb{R}^n$. Defining the half–spaces

$$S_k = \left\{ (w, r) \in \mathbb{R}^{n+1} \;\middle|\; r \in \mathbb{R} \text{ and } \langle (w - z_k), e_k \rangle_{\mathbb{R}^n} \leq 0 \right\},$$

we conclude that $\overline{U_k} \subset S_k$ for each k. Modulo picking a subsequence, we may without loss of generality assume that there is a unit vector $e \in S^{n-1}$ and a point $x = (z, 0) \in K$ such that $e_k \overset{k\to\infty}{\longrightarrow} e$ and $z_k \overset{k\to\infty}{\longrightarrow} z$ (S^{n-1} and K are compact). Then, since the $\overline{U_k}$ converge to V, it follows that

$$V \subset \left\{ (w, 0) \in \mathbb{R}^{n+1} \;\middle|\; \langle (w - z), e \rangle_{\mathbb{R}^n} \leq 0 \right\}.$$

But since $x = (z, 0) \in K \subset I$, there is an $\eta > 0$ such that $B_\eta(x) \cap \pi_0^1 \subset V$, which contradicts the inclusion above (e.g., $\big(z + \frac{\eta}{2} e, 0\big) \in V$, but $\left\langle \Big(\big(z + \frac{\eta}{2} e\big) - z \Big), e \right\rangle_{\mathbb{R}^n} = \frac{\eta}{2} > 0$). $\square$

Now let $K \subset I$ be compact. For all $x = (z, 0) \in K$, let

$$\nu_k^+(x) \quad \text{be the outer unit normal to } \partial U_k \text{ in } \big(z, b(z)\big)$$

and

$$\nu_k^-(x) \quad \text{be the outer unit normal to } \partial U_k \text{ in } \big(z, a(z)\big).$$

Then we have

Claim 3.

$$\lim_{k\to\infty} \max_{x \in K} \Big\{ \big| \nu_k^+(x) - (0, \ldots, 0, 1) \big| + \big| \nu_k^-(x) - (0, \ldots, 0, -1) \big| \Big\} = 0.$$

PROOF. We restrict ourselves to showing that

$$\lim_{k\to\infty} \max_{x\in K}\Big\{\big|\nu_k^+(x) - (0,\dots,0,1)\big|\Big\} = 0,$$

the other limit following completely analogously. For each $k \in \mathbb{N}$, let $x_k = (z_k, 0) \in K$ be such that

$$\big|\nu_k^+(x_k) - (0,\dots,0,1)\big| = \max_{x\in K}\Big\{\big|\nu_k^+(x) - (0,\dots,0,1)\big|\Big\}$$

(K is compact). By the convexity of U_k, we have that

$$\overline{U_k} \subset T_k = \Big\{y \in \mathbb{R}^{n+1} \ \Big|\ \big\langle y - \big(z_k, b(z_k)\big), \nu_k^+(x_k)\big\rangle \le 0\Big\}.$$

But since $\big(z_k, a(z_k)\big) \in \overline{U_k}$ and $a(z_k) < b(z_k)$, this implies that

$$\big\langle \nu_k^+(x_k), (0,\dots,0,1)\big\rangle \ge 0 \qquad \forall k \in \mathbb{N}. \tag{2.4}$$

Now assume ν^+ is the limit of a subsequence of $\big(\nu_k^+(z_k)\big)_{k\in\mathbb{N}}$ (S^{n-1} is compact), and $x = (z,0) \in K$ is the limit of a further subsequence of $(x_k)_{k\in\mathbb{N}} = \big((z_k,0)\big)_{k\in\mathbb{N}}$ (K is compact). It then follows that

$$V \subset T = \Big\{y \in \mathbb{R}^{n+1} \ \Big|\ \big\langle y - (z,0), \nu^+\big\rangle \le 0\Big\}.$$

But, since $(z,0) \in K \subset I$, there is a $\eta > 0$ such that $\big(z + \eta e, 0\big) \in I$, $\forall e \in S^{n-1}$. Consequently, ν^+ must be orthogonal to π_0^1, from which

$$\textit{either} \qquad \nu^+ = (0,\dots,0,1) \qquad \textit{or} \qquad \nu^+ = (0,\dots,0,-1).$$

(2.4) then yields $\nu^+ = (0,\dots,0,1)$, which is precisely what we wanted to show. □

Now consider, for every $\epsilon \in (0,1)$, the sets

$$\begin{aligned} B_\epsilon &= \Big\{\big((1-\epsilon)z, 0\big) \ \Big|\ (z,0) \in B\Big\}, \\ I_\epsilon &= \Big\{(\rho z, 0) \ \Big|\ (z,0) \in B, 0 \le \rho \le 1-\epsilon\Big\} \end{aligned}$$

and

$$C_\epsilon = \Big\{(z,r) \ \Big|\ (z,0) \in I_\epsilon, r \in \mathbb{R}\Big\}$$

(remember that we had assumed $0 \in V$). Set

$$\Sigma_{k,\epsilon}^i = \partial U_k \cap C_\epsilon \quad \text{and} \quad \Sigma_{k,\epsilon}^e = \partial U_k \setminus \Sigma_{k,\epsilon}^i \qquad (\epsilon \in (0,1),\ k \in \mathbb{N}).$$

Then we have

Claim 4.

$$\lim_{\epsilon\searrow 0} \lim_{k\to\infty} \mathrm{vol}_n\left(\Sigma_{k,\epsilon}^e\right) = 0.$$

Proof. From claims 2 and 3, we conclude that

$$\lim_{k\to\infty} \mathrm{vol}_n\left(\Sigma^i_{k,\epsilon}\right) = 2\mathrm{vol}_n\left(I_\epsilon\right). \tag{2.5}$$

This can be seen as follows. Claim 2 tells us that $\Sigma^i_{k,\epsilon}$ is the union of two graphs over the compact set $I_\epsilon \subset I$, namely those of the functions $a(z)$ and $b(z)$ ($(z,0) \in I_\epsilon$). By Claim 3, both of these functions converge, as $k \to \infty$, to a constant (in fact, both converge to 0) on all of I_ϵ. This implies the above assertion.
Considering then, as in the proof of Claim 1, the tubular neighbourhood $V_\delta = \left\{x \in \mathbb{R}^{n+1} \,\middle|\, \mathrm{dist}(x,V) < \delta\right\}$ $(\delta > 0)$ of V, we know that $U_k \subset V_\delta$ for k large enough (with respect to δ), whence (U_k is convex)

$$\mathrm{vol}_n\left(\partial U_k\right) \leq \mathrm{vol}_n\left(\partial V_\delta\right) \qquad (k \text{ large enough}).$$

But since we also have

$$\lim_{\delta\searrow 0} \mathrm{vol}_n\left(\partial V_\delta\right) = 2\mathrm{vol}_n(V),$$

we conclude that

$$1 = \limsup_{k\to\infty} \underbrace{\mathrm{vol}_n\left(\partial U_k\right)}_{=1} \leq 2\mathrm{vol}_n(V). \tag{2.6}$$

Taking into account that $\lim_{\epsilon\searrow 0} \mathrm{vol}_n\left(I_\epsilon\right) = \mathrm{vol}_n(V)$, as well as that $\mathrm{vol}_n\left(\Sigma^i_{k,\epsilon}\right) \leq \mathrm{vol}_n\left(\partial U_k\right) = 1$, the combination of (2.6) with (2.5) yields, on one hand, that

$$\mathrm{vol}_n(V) = \frac{1}{2}$$

and, on the other hand, that

$$\lim_{\epsilon\searrow 0} \lim_{k\to\infty} \mathrm{vol}_n\left(\Sigma^i_{k,\epsilon}\right) = 1,$$

from which the claim follows. □

Fix $\epsilon \in (0,1)$ and $k \in \mathbb{N}$. For every $x = (z,0) \in B_\epsilon$, pick a unit normal $\nu(x)$ to B_ϵ in π^1_0.

Remark 2.12. *Of course, B_ϵ might not be smooth, but since it is the boundary (in π^1_0) of a convex set (I_ϵ), we know from [**Roc70**, Thm.25.5, p.246], that $\nu(x)$ will be uniquely defined except for a set of zero $(n-1)$–dimensional Hausdorff measure.*

Consider, for each $x = (z,0) \in B_\epsilon$, the two–dimensional half–plane

$$\tau^+(x) = \left\{x + (0,\ldots,0,s) + t\nu(x) \,\middle|\, s,t \geq 0\right\}.$$

Then the intersection

$$\gamma_{k,x} = \tau^+(x) \cap \partial U_k$$

of $\tau^+(x)$ with ∂U_k is, by Claim 2, a curve in $\tau^+(x)$ joining $\big(z,a(z)\big)$ with $\big(z,b(z)\big)$ $(x = (z,0))$. Thus, by the coarea formula (see, again, [**Cha06**, §III.8] or [**Fed69**,

§3.2]), and taking into account Remark 2.12, we have

$$\liminf_{k\to\infty}\int_{\Sigma^e_{k,\epsilon}}|A| = \liminf_{k\to\infty}\int_{B_\epsilon}\left(\int_{\gamma_{k,x}}|A|\right)dx \ge \liminf_{k\to\infty}\int_{B_\epsilon}\left|\nu_k^+(x)-\nu_k^-(x)\right|dx$$
$$\ge \mathrm{vol}_{n-1}(B_\epsilon) = (1-\epsilon)^{n-1}\mathrm{vol}_{n-1}(B),$$

where the first inequality follows from Corollary A.4 of the appendix, whereas the second is a consequence of Claim 3. We conclude

$$\liminf_{\epsilon\searrow 0}\liminf_{k\to\infty}\int_{\Sigma^e_{k,\epsilon}}|A| \ge \mathrm{vol}_{n-1}(B) > 0. \tag{2.7}$$

On the other hand, given that $\int_{\partial U_k}|A|^p \le c_0$ and $p>1$ by assumption, Hölder's inequality yields

$$\int_{\Sigma^e_{k,\epsilon}}|A| \le \left(\mathrm{vol}_{n-1}\left(\Sigma^e_{k,\epsilon}\right)\right)^{1-\frac{1}{p}}\left(\int_{\Sigma^e_{k,\epsilon}}|A|^p\right)^{\frac{1}{p}} \le c_0^{\frac{1}{p}}\left(\mathrm{vol}_{n-1}\left(\Sigma^e_{k,\epsilon}\right)\right)^{1-\frac{1}{p}}.$$

Using Claim 4, we then conclude

$$\limsup_{\epsilon\searrow 0}\limsup_{k\to\infty}\int_{\Sigma^e_{k,\epsilon}}|A| = 0,$$

which contradicts (2.7). Our assumption at the beginning of the proof must therefore be wrong, and the lemma is proved. □

CHAPTER 3

The L^2–theory

In this chapter we prove our main estimate in the L^2–case for n–dimensional hypersurfaces of $\mathbb{R}^{n+1}$ with non–negative Ricci curvature (which is equivalent to being convex). The method thereby used mimics an argument in [**DLT10**]. Afterwards, we give an alternative proof due to G. Huisken of this estimate in the two–dimensional case under the assumption that the surface is mean convex and constitutes the boundary of a star–shaped domain. Finally, we exhibit how that last proof lends itself to generalisation to the n–dimensional case.

Contents

1. The case Ric ≥ 0

We first prove the L^2–estimate for hypersurfaces that have non–negative Ricci curvature, with the constant $C = \sqrt{\frac{n}{n-1}}$ on the right–hand side. As we shall see in the next chapter, that constant is optimal. There, we will also prove that the assumption Ric ≥ 0 is optimal whenever $n \geq 3$ (i.e. in the sub–critical case).

Theorem 3.1. *Let $n \geq 2$ be given and set $C = \sqrt{\frac{n}{n-1}}$. Then we have: if Σ is a smooth, closed, connected hypersurface in $\mathbb{R}^{n+1}$ with induced Riemannian metric g and non–negative Ricci curvature, then*

$$\left(\int_\Sigma \left| A - \frac{1}{n}\left(\frac{1}{\mathrm{vol}_n(\Sigma)}\int_\Sigma H\right) g\right|^2\right)^{\frac{1}{2}} \leq C\left(\int_\Sigma \left|A - \frac{H}{n} g\right|^2\right)^{\frac{1}{2}}. \tag{3.1}$$

In particular, the above estimate holds for smooth, closed hypersurfaces which are the boundary of a convex set in $\mathbb{R}^{n+1}$.

Proof. The following argument is an adaptation of the proof of Theorem 0.1 in [**DLT10**] (see also [**CLN06**, §B.3, pp.517–519]).

Let

$$\mathring{A} = A - \frac{1}{n} H g \qquad \text{and} \qquad \overline{H} = \frac{1}{\mathrm{vol}_n(\Sigma)} \int_\Sigma H,$$

and write the square of the left–hand side of (3.1) as

$$\begin{aligned}
\int_\Sigma \left| A - \frac{\overline{H}}{n} g \right|^2 &= \int_\Sigma \left| \left(A - \frac{H}{n} g \right) + \left(\frac{H - \overline{H}}{n} g \right) \right|^2 \\
&= \int_\Sigma |\mathring{A}|^2 + \frac{2}{n} \int_\Sigma \left(H - \overline{H} \right) \sum_{i,j=1}^n g^{ij} \mathring{A}_{ij} + \frac{1}{n} \int_\Sigma \left| H - \overline{H} \right|^2 \\
&= \int_\Sigma |\mathring{A}|^2 + \frac{1}{n} \int_\Sigma \left| H - \overline{H} \right|^2 .
\end{aligned} \tag{3.2}$$

Let φ be the unique smooth solution of the following Poisson problem on Σ (for existence, uniqueness and regularity see, e.g., [**Aub98**, Thm.4.7, p.104]):

$$\begin{cases} \Delta\varphi &= H - \overline{H}, \\ \int_\Sigma \varphi &= 0. \end{cases}$$

We then have

$$\int_\Sigma \left| H - \overline{H} \right|^2 = \int_\Sigma \left(H - \overline{H} \right) \Delta\varphi = \int_\Sigma \left(H - \overline{H} \right) \sum_{i=1}^n \nabla_i \nabla^i \varphi = - \int_\Sigma \sum_{i=1}^n \nabla_i H \nabla^i \varphi.$$

By virtue of the Codazzi equations we find

$$\nabla_i H = \sum_{l=1}^n \nabla_i A^l{}_l = \sum_{l=1}^n \nabla_l A^l{}_i = \sum_{l=1}^n \nabla_l \mathring{A}^l{}_i + \frac{1}{n} \sum_{l=1}^n \nabla_l H \delta^l{}_i = \frac{n}{n-1} \sum_{l=1}^n \nabla_l \mathring{A}^l{}_i.$$

Thus

$$\int_\Sigma \left| H - \overline{H} \right|^2 = - \frac{n}{n-1} \int_\Sigma \sum_{i,l=1}^n \nabla_l \mathring{A}^l{}_i \nabla^i \varphi = \frac{n}{n-1} \int_\Sigma \mathring{A} \colon \mathrm{Hess}\, \varphi.$$

Since $\mathring{A}$ is trace–free, we have

$$\mathring{A} \colon \mathrm{Hess}\, \varphi = \mathring{A} \colon \left(\mathrm{Hess}\, \varphi - \frac{1}{n} \Delta\varphi g \right),$$

and thus, by Cauchy–Schwarz,

$$\begin{aligned}
\int_\Sigma \left|H-\overline{H}\right|^2 &= \frac{n}{n-1}\int_\Sigma \mathring{A}\colon\left(\operatorname{Hess}\varphi-\frac{1}{n}\Delta\varphi g\right)\\
&\leq \frac{n}{n-1}\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}\left(\int_\Sigma \left|\operatorname{Hess}\varphi-\frac{1}{n}\Delta\varphi g\right|^2\right)^{\frac{1}{2}}\\
&= \frac{n}{n-1}\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |\operatorname{Hess}\varphi|^2-\frac{1}{n}\int_\Sigma |\Delta\varphi|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
(3.3)\qquad \int_\Sigma |\operatorname{Hess}\varphi|^2 &= \int_\Sigma \sum_{i,j=1}^n \nabla^i\nabla^j\varphi\nabla_i\nabla_j\varphi = -\int_\Sigma \sum_{i,j=1}^n \nabla^j\varphi\nabla^i\nabla_i\nabla_j\varphi\\
&= -\int_\Sigma \sum_{i,j=1}^n \nabla^j\varphi\nabla^i\nabla_j\nabla_i\varphi\\
&= -\int_\Sigma \sum_{i,j=1}^n \nabla^j\varphi\nabla_j\nabla^i\nabla_i\varphi - \int_\Sigma \sum_{i,j=1}^n \nabla^j\varphi\operatorname{Ric}_j{}^i\,\nabla_i\varphi\\
&= \int_\Sigma |\Delta\varphi|^2 - \int_\Sigma \operatorname{Ric}(\nabla\varphi,\nabla\varphi)\,,
\end{aligned}$$

we have

$$\begin{aligned}
\int_\Sigma \left|H-\overline{H}\right|^2 &\leq \frac{n}{n-1}\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}\left(\frac{n-1}{n}\int_\Sigma |\Delta\varphi|^2-\int_\Sigma \operatorname{Ric}(\nabla\varphi,\nabla\varphi)\right)^{\frac{1}{2}}\\
&\leq \frac{n}{n-1}\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}\left(\frac{n-1}{n}\int_\Sigma |\Delta\varphi|^2\right)^{\frac{1}{2}}\\
&= \left(\frac{n}{n-1}\right)^{\frac{1}{2}}\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |\Delta\varphi|^2\right)^{\frac{1}{2}}\\
&= \left(\frac{n}{n-1}\right)^{\frac{1}{2}}\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}\left(\int_\Sigma \left|H-\overline{H}\right|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used the assumption Ric ≥ 0. Therefore

$$\int_\Sigma \left|H-\overline{H}\right|^2 \leq \frac{n}{n-1}\int_\Sigma |\mathring{A}|^2\,,$$

and so

$$\int_\Sigma \left|A-\frac{\overline{H}}{n}g\right|^2 \leq \int_\Sigma |\mathring{A}|^2+\frac{1}{n-1}\int_\Sigma |\mathring{A}|^2 = \frac{n}{n-1}\int_\Sigma |\mathring{A}|^2\,,$$

as claimed. ☐

2. Why Ric ≥ 0 and convexity are the same

In this section we present an argument, recently brought to the author's attention by C. De Lellis, which concludes that every smooth, closed and connected hypersurface of $\mathbb{R}^{n+1}$ with non–negative Ricci curvature must be convex. Although this seems to be a well–known fact (cf., e.g., [**Des92**]), we could not yet find a proof in the literature, and expose C. De Lellis' argument for the sake of completeness.

Proposition 3.2. *Let $n \geq 2$ and suppose Σ a smooth, closed and connected hypersurface of $\mathbb{R}^{n+1}$. Then the following are equivalent:*

(i) Ric ≥ 0 *everywhere on* Σ*;*
(ii) $A \geq 0$ *everywhere on* Σ.

In particular, if either of the above conditions hold, Σ is convex.

Proof (by C. De Lellis). Denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of A. In view of the contracted Gauss equations, $\mathrm{Ric} = HA - A^2$, it is immediate to see that condition (ii) implies condition (i). The interesting question is therefore the converse implication.

We first argue on a pointwise level. Ric ≥ 0 implies that, for every $i \in \{1, \dots, n\}$, we have

$$\left(\sum_{j=1}^{n} \lambda_j\right) \lambda_i \geq \lambda_i^2 \geq 0.$$

Thus, each principal curvature λ_i needs to be of the same sign than the mean curvature H. We conclude that, in each point of the hypersurface, Ric ≥ 0 implies semi–definiteness of A. In other words, for each $q \in \Sigma$, one of the following three situations hold:

(a) $A(q) > 0$,

(b) $A(q) < 0$

or

(c) $A(q) = 0$.

We now want to obtain the global statement by arguing that A cannot change sign. Suppose this were not true, i.e., assume there were points $q_+ \in \Sigma$ and $q_- \in \Sigma$, such that $A(q_-) < 0 < A(q_+)$. Clearly, by continuity, it follows that

$$\mathrm{vol}_n\left(U_+\right) > 0, \qquad \text{where } U_+ = \left\{q \in \Sigma \;\middle|\; A(q) > 0\right\} \subset \Sigma,$$

and

$$\mathrm{vol}_n\left(U_-\right) > 0, \qquad \text{where } U_- = \left\{q \in \Sigma \;\middle|\; A(q) < 0\right\} \subset \Sigma.$$

For $e \in S^n$ and $c \in \mathbb{R}$, consider the hyperplane

$$T_e(c) = \left\{x \in \mathbb{R}^{n+1} \;\middle|\; \langle x, e\rangle = c\right\}.$$

Furthermore, let

$$c_e = \min\left\{c \in \mathbb{R} \;\middle|\; T_e(c) \cap \Sigma \neq \emptyset\right\}$$

denote the smallest value $c \in \mathbb{R}$ for which $T_e(c)$ intersects Σ. Clearly, Σ will then lie on one side of $T_e(c_e)$, and $\Sigma \cap T_e(c_e) \neq \emptyset$. Geometrically, $T_e(c_e)$ is a supporting hyperplane of Σ, and we must have that $A(q) \geq 0$, whenever $q \in T_e(c_e) \cap \Sigma$. Since e was arbitrary, we conclude

$$\nu\left(\left\{q \in \Sigma \;\middle|\; A(q) \geq 0\right\}\right) = S^n.$$

But, since for all $q \in \Sigma$, we have $T_q\Sigma = T_{\nu(q)}S^n$, the maps $A(\,\cdot\,,\,\cdot\,)$ and $\big\langle d\nu(\,\cdot\,),\,\cdot\,\big\rangle$ are equal, and Sard's theorem (see, e.g., [**DFN85**, Thm.10.2.1, p.79]) applied to $\nu : \Sigma \to S^n$ ensures that

$$\mathrm{vol}_n\Big(\nu\big(\big\{q \in \Sigma \;\big|\; A(q) = 0\big\}\big)\Big) = 0,$$

whence

$$\nu(U_+) = S^n \setminus N,$$

for some null–set $N \subset S^n$.

Moreover, since almost every value of ν is regular and Σ is compact, the number $\mathrm{vol}_0\big(\nu^{-1}(\,\cdot\,)\big)$ of pre–images under ν is finite for almost every point in S^n, as well as locally constant. In view of the area formula (see, e.g., [**Fed69**, §3.2]),

$$0 < \int_{U_-} |\det A| = \int_{\nu(U_-)} \mathrm{vol}_0\left(\nu^{-1}(\,\cdot\,) \cap U_-\right),$$

we then conclude that

$$\mathrm{vol}_n\big(\nu(U_-)\big) > 0.$$

From these considerations follows that the set

$$B = \Big\{e \in S^n \;\Big|\; \exists q_+ \in U_+, q_- \in U_- \text{ such that } \nu(q_+) = \nu(q_-) = e\Big\} \subset S^n$$

has strictly positive measure. Now, for $e \in S^n$, consider the map $f_e : \Sigma \to \mathbb{R}$, $q \mapsto \langle q, e\rangle$. Then the critical points of f_e are given by the set

$$C_e = \Big\{q \in \Sigma \;\Big|\; \nu(q) = \pm e\Big\} \subset \Sigma.$$

We show that f_e is a Morse function for almost every $e \in S^n$. To do this, we have to check that $\mathrm{Hess}\, f_e(q)$ has full rank whenever $q \in C_e$. But there, we have

$$\mathrm{Hess}\, f_e(q) = \mp A(q) \qquad (q \in C_e),$$

and we conclude that f_e is a Morse function whenever

$$C_e \subset \Big\{q \in \Sigma \;\Big|\; \det A(q) \neq 0\Big\},$$

i.e., whenever e and $-e$ are regular values of ν. But since the set of singular values of ν has measure zero, we conclude that, for almost all $e \in S^n$, the map f_e is a Morse function. In particular, since $\mathrm{vol}_n(B) > 0$, there is an $\overline{e} \in B$ for which $f_{\overline{e}}$ is Morse, and we get by construction

$$\exists \overline{q}_+ \in \Sigma \quad \text{such that} \quad \nu(\overline{q}_+) = \overline{e} \quad \text{and} \quad \mathrm{Hess}\, f_{\overline{e}}(\overline{q}_+) > 0,$$

and

$$\exists \overline{q}_- \in \Sigma \quad \text{such that} \quad \nu(\overline{q}_-) = \overline{e} \quad \text{and} \quad \mathrm{Hess}\, f_{\overline{e}}(\overline{q}_-) < 0.$$

On the other hand, we saw that $f_{\overline{e}}$ must have the absolute minimum $c_{\overline{e}}$ in a point $\overline{q}_0 \in T_{\overline{e}}(c_{\overline{e}}) \cap \Sigma$ for which we have $\operatorname{Hess} f_{\overline{e}}(\overline{q}_0) > 0$. Moreover, $\nu(\overline{q}_0) = -\overline{e}$, whence $\overline{q}_0 \neq \overline{q}_+$. By construction, then, $f_{\overline{e}}$ has two distinct local minima.

But since Σ is connected, one of the Morse inequalities yields (see, e.g., [**DFN90**, §16] or [**Nic07a**, §2.3])

$$\mu_1 - 2 \geq \mu_1 - \mu_0 \geq b_1 - b_0 = b_1 - 1,$$

from which $\mu_1 \geq 1$. Here, b_i and μ_i denote the Betti numbers and the Morse numbers, respectively, and we have used that $b_1 \geq 0$, $b_0 = 1$ (by connectedness) and $\mu_0 \geq 2$ (since $f_{\overline{e}}$ has, at least, two local minima). Consequently, $f_{\overline{e}}$ needs to have at least one saddle point of index 1. This, however, is impossible, since all the critical points of $f_{\overline{e}}$ have either index 0 (minima) or index n (maxima), by construction. Thus our assumption that there is a point in which $A < 0$ was false, and the proposition is proved. □

3. G. Huisken's proof for two–dimensional mean–convex surfaces that bound a star–shaped domain

In this section we prove the L^2–estimate (3.1) for two–dimensional surfaces that are mean convex and bound a star–shaped domain in $\mathbb{R}^3$. The method hereby used was suggested by G. Huisken and uses inverse mean curvature flow. The additional assumptions recover the optimal constant $C = \sqrt{2}$ (cf. Proposition 4.1 of the next chapter). However, as we are also going to show in Chapter 4, the constant $C = \sqrt{2}$ does not work for generic surfaces (compare with [**DLM05**]). We wish to stress here that the following proof takes care of a more general situation than the corresponding result in Section 1 ($H > 0$ and star–shaped *versus* $\operatorname{Ric} \geq 0$, i.e. convex), except for the fact that our requirement is weak (non–strict inequality). However, it seems an easy, albeit tedious matter to generalise Theorem 3.3 below to the case $H \geq 0$ by approximation with ($H > 0$)–surfaces (compare also with [**HI08**, Theorem 2.5] for weakening of $H > 0$ to $H \geq 0$ in the case of strict star–shapedness).

Theorem 3.3 (Huisken)**.** *If Σ is the smooth, closed boundary of a star–shaped domain in $\mathbb{R}^3$ with induced Riemannian metric g, and has everywhere strictly positive mean curvature H, then*

$$\left(\int_\Sigma \left| A - \frac{1}{2} \left(\frac{1}{\operatorname{vol}_2(\Sigma)} \int_\Sigma H \right) g \right|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \left(\int_\Sigma \left| A - \frac{H}{2} g \right|^2 \right)^{\frac{1}{2}}. \tag{3.4}$$

The proof presented below differs only slightly from the one proposed by G. Huisken to C. De Lellis at a summer school in Rome (Italy) in 2005.

3.1. Preliminaries. For M an n–dimensional, smooth, closed manifold and $T > 0$, a solution to the inverse mean curvature flow (IMCF) is given by a smooth family of embeddings $F : M \times [0, T] \to \mathbb{R}^{n+1}$, such that

$$\partial_t F(x,t) = \frac{1}{H(x,t)} \nu(x,t), \qquad x \in M, \quad t \in [0,T],$$

where $H(x,t) > 0$ and $\nu(x,t)$ are the mean curvature and the exterior unit normal of $F(M,t)$ in the point $F(x,t)$. In 1990, Gerhardt [**Ger90**] and Urbas [**Urb90**] proved independently that, if the initial data $F(M,0) = \Sigma$ is the smooth boundary of a star–shaped domain, then the IMCF has a smooth solution for all times $t > 0$ which approaches a homothetically expanding spherical solution as $t \to +\infty$.

Consequently, the following approach to proving Theorem 3.3 seems promising. Consider the functional

$$\mathcal{F}(\Sigma) = \int_\Sigma |\mathring{A}|^2 - \frac{1}{2}\int_\Sigma \left(H - \frac{1}{\mathrm{vol}_2(\Sigma)}\int_\Sigma H\right)^2$$

on the set of smooth, closed surfaces of $\mathbb{R}^3$. Notice that it is scale–invariant, and that the positivity of $\mathcal{F}(\Sigma)$ is equivalent to inequality (3.4). Also, $\mathcal{F}(S^2) = 0$. In view of the results about IMCF mentioned above, it is then sufficient to prove that $\mathcal{F}$ is non–increasing along the flow starting at the surface Σ which bounds a star–shaped domain.

3.2. Proof of Theorem 3.3. If Σ is the smooth, closed boundary of a star–shaped (with respect to, say, the origin) domain in $\mathbb{R}^3$ with $H > 0$, let Σ_t denote the smooth, global solution constructed in [**Ger90**] or [**Urb90**] of the IMCF starting at $\Sigma_0 = \Sigma$. Then the rescaled surfaces $\frac{4\pi}{\mathrm{vol}_2(\Sigma_t)}\Sigma_t$ converge to the round sphere $S^2(0)$ as $t \to +\infty$. If we introduce, for any smooth function $\varphi : \Sigma \times [0,+\infty) \to \mathbb{R}$, the notation

$$\overline{\varphi} = \frac{1}{\mathrm{vol}_2(\Sigma_t)}\int_{\Sigma_t} \varphi,$$

then the following lemma immediately implies the theorem.

Lemma 3.4. *For all $t \geq 0$ we have*

$$\frac{d}{dt}\mathcal{F}(\Sigma_t) = -\overline{H}\int_{\Sigma_t} \frac{|\mathring{A}|^2}{H}.$$

Indeed, $\mathcal{F}(\Sigma_t)$ is then non–increasing along the IMCF. And since $\mathcal{F}$ is scale–invariant and $\lim_{t\to+\infty}\mathcal{F}(\Sigma_t) = \mathcal{F}(S^2) = 0$, we conclude that $\mathcal{F}(\Sigma_t) \geq 0$ for all $t \geq 0$ and, in particular, that $\mathcal{F}(\Sigma) \geq 0$. This proves Theorem 3.3. $\square$

3.3. Proof of Lemma 3.4. We wish to remark that the functional $\mathcal{F}$ considered here is a special case of the functional given by (4.2), to be studied in Section 1 of Chapter 4. The calculations for its first variation performed there are, of course, valid also in the setting at hand, and we recover from (4.3s) with $f = 1/H$,

$n = 2$ and $C = \sqrt{2}$ that, for all $t \geq 0$,

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}(\Sigma) &= \int_\Sigma |\mathring{A}|^2 - 2\int_\Sigma \operatorname{Hess}\frac{1}{H}{:}\mathring{A} - 2\int_\Sigma \frac{A^2{:}\mathring{A}}{H} \\
&\quad - \frac{1}{2}\int_\Sigma \left(H - \overline{H}\right)^2 + \int_\Sigma \left(H - \overline{H}\right)\Delta\frac{1}{H} + \int_\Sigma |A|^2\,\frac{H - \overline{H}}{H} \\
&= \int_\Sigma |\mathring{A}|^2 - 2\int_\Sigma \operatorname{Hess}\frac{1}{H}{:}A + \int_\Sigma H\Delta\frac{1}{H} - 2\int_\Sigma \frac{A^2{:}\mathring{A}}{H} \\
&\quad - \frac{1}{2}\int_\Sigma H^2 + \frac{1}{2}\overline{H}\int_\Sigma H + \int_\Sigma H\Delta\frac{1}{H} \\
&\quad - \overline{H}\int_\Sigma \Delta\frac{1}{H} + \int_\Sigma |A|^2 - \overline{H}\int_\Sigma \frac{|A|^2}{H} \\
&= 2\int_\Sigma |\mathring{A}|^2 - 2\int_\Sigma \frac{A^2{:}\mathring{A}}{H} - \overline{H}\int_\Sigma \frac{|\mathring{A}|^2}{H} \\
&\quad - 2\int_\Sigma \operatorname{Hess}\frac{1}{H}{:}A + 2\int_\Sigma H\Delta\frac{1}{H} - \overline{H}\int_\Sigma \Delta\frac{1}{H}.
\end{aligned}
\tag{3.5}
$$

Now, since Σ is closed, $\int_\Sigma \Delta\varphi$ vanishes for any C^2–function φ on Σ. Also, in view of the Codazzi equations, we have

$$
\begin{aligned}
\int_\Sigma \operatorname{Hess}\varphi{:}A &= \int_\Sigma \sum_{i,j=1}^{2} \nabla_i\nabla^j\varphi A^i{}_j = -\int_\Sigma \sum_{i,j=1}^{2} \nabla^j\varphi\nabla_i A^i{}_j \\
&= -\int_\Sigma \sum_{i,j=1}^{2} \nabla^j\varphi\nabla_j A^i{}_i = \int_\Sigma \sum_{i,j=1}^{2} \nabla_j\nabla^j\varphi A^i{}_i = \int_\Sigma H\Delta\varphi.
\end{aligned}
$$

Finally, recalling that every two–dimensional Riemannian manifold is Einstein (i.e. its Ricci curvature is a multiple of its metric, $\operatorname{Ric} = \frac{\operatorname{Scal}}{2}g$), we see that $\operatorname{Ric}{:}\mathring{A} = 0$. But since the (once contracted) Gauss equations tell us that $\operatorname{Ric} = HA - A^2$, we conclude that

$$
\frac{A^2{:}\mathring{A}}{H} = A{:}\mathring{A} = |\mathring{A}|^2\,.
$$

Putting these three observations together simplifies (3.5) to

$$
\frac{d}{dt}\mathcal{F}(\Sigma) = -\overline{H}\int_\Sigma \frac{|\mathring{A}|^2}{H},
$$

as required. □

4. The flow approach to n dimensions

The purpose of this section is to investigate how G. Huisken's method explained in Section 3 can be generalised to the n–dimensional case. As it turns out, requiring merely $H > 0$ for the boundary of a star–shaped domain is not enough, but we have to assume that the domain is strictly convex. Moreover, we cannot recover the optimal constant $C = \sqrt{\frac{n}{n-1}}$ (cf. Proposition 4.1), but only get the result for

$C = \sqrt{n}$, and the reason for this remains unclear until now. Otherwise, the method generalises quite straightforwardly, although a much more careful inspection of the rate of change of the considered functional is necessary.

4.1. The setup. Our goal will be to prove

Theorem 3.5. *Let $n \geq 3$ be given. Then we have: if Σ is the smooth, closed boundary of a strictly convex domain in $\mathbb{R}^{n+1}$ with induced Riemannian metric g, then*

$$\left(\int_\Sigma \left| A - \frac{1}{n}\left(\frac{1}{\mathrm{vol}_n(\Sigma)}\int_\Sigma H\right) g \right|^2\right)^{\frac{1}{2}} \leq \sqrt{n}\left(\int_\Sigma \left| A - \frac{H}{n} g\right|^2\right)^{\frac{1}{2}}. \tag{3.6}$$

According to [**Urb91**, Theorem 1.1], the inverse mean curvature flow with Σ as initial data has a smooth, global (i.e. which exists for all $t \geq 0$) solution Σ_t which converges (as $t \to +\infty$) to a round sphere after rescaling to constant volume. As in the previous section, we want to consider a scale–invariant functional which vanishes on spheres and represents the sought–after inequality (3.6). We then show that this functional is monotone along the IMCF.

More precisely, consider the functional

$$\mathcal{F}(\Sigma) = (n-1)\int_\Sigma |\mathring{A}|^2 - \frac{1}{n}\int_\Sigma \left(H - \overline{H}\right)^2$$

on the set of smooth, closed hypersurfaces of $\mathbb{R}^{n+1}$, where, again, $\overline{H} = \frac{1}{\mathrm{vol}_n(\Sigma)}\int_\Sigma H$. It is immediate to see that $\mathcal{F}(\Sigma)$ is non–negative if and only if inequality (3.6) holds (use (3.2)). Notice also that $\mathcal{F}(S^n) = 0$. We rescale $\mathcal{F}$ to the scale–invariant quantity

$$\mathcal{H}(\Sigma) = \mathrm{vol}_n^{-\frac{n-2}{n}}(\Sigma)\mathcal{F}(\Sigma) = \mathrm{vol}_n^{-\frac{n-2}{n}}(\Sigma)\left((n-1)\int_\Sigma |\mathring{A}|^2 - \frac{1}{n}\int_\Sigma (H - \overline{H})^2\right).$$

By the same arguments as in the proof of Theorem 3.3, the following proposition then immediately implies Theorem 3.5.

Proposition 3.6. *For Σ, Σ_t an $\mathcal{H}$ as above, we have*

$$\frac{d}{dt}\mathcal{H}(\Sigma_t) \leq 0.$$

Proof of Proposition 3.6. In the next subsection, we shall show the following lemma giving the rate of change of $\mathcal{H}(\Sigma_t)$ along the IMCF.

Lemma 3.7. *For Σ, Σ_t and $\mathcal{H}$ as above, we have*

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(\Sigma_t) = -2\mathrm{vol}_n^{-\frac{n-2}{n}}(\Sigma_t)\Bigg(&\frac{1}{n}\overline{H}\int_{\Sigma_t}\frac{|\mathring{A}|}{H} + (n-2)\int_{\Sigma_t}\frac{|\nabla H|^2}{H^2} \\ &+ (n-1)\int_{\Sigma_t}\frac{\mathrm{tr}_g A^3}{H} - \frac{2n-1}{n}\int_{\Sigma_t}|A|^2 + \frac{1}{n}\int_{\Sigma_t} H^2\Bigg).\end{aligned}$$

The first two terms in parentheses in the above expression being obviously non–negative, we have to focus our attention on the three terms on the second line to see that $\mathcal{H}(\Sigma_t)$ is non–increasing. If we denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A (i.e. the principal curvatures of Σ_t), and introduce

$$\mu_i = \frac{\lambda_i}{H} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j} \qquad (i \in \{1, \ldots, n\}),$$

then the sign of $\frac{d}{dt}\mathcal{H}(\Sigma_t)$ immediately follows from the lemma below (the proof of which will be performed in Subsection 4.3), since it implies

$$(n-1)\frac{\operatorname{tr}_g A^3}{H} - \frac{2n-1}{n}|A|^2 + \frac{1}{n}H^2 \geq 0.$$

Lemma 3.8. *If $n \geq 3$, then the function*

$$g(\mu) = (n-1)\sum_{i=1}^n \mu_i^3 - \frac{2n-1}{n}\sum_{i=1}^n \mu_i^2 + \frac{1}{n}$$

is non–negative on the domain $\pi = \left\{\mu \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n \mu_i = 1, \mu_i \geq 0\right\}$ and vanishes only if all the μ_i equal $\frac{1}{n}$ or one μ_i vanishes whereas the others all equal $\frac{1}{n-1}$.

Clearly, then, Lemma 3.7 and Lemma 3.8 imply the proposition. □

4.2. Proof of Lemma 3.7. As in the proof of Lemma 3.4, the calculations of Section 1, Chapter 4, can be used also here. From (4.3c) (with $f = 1/H$) we recover the well–known fact that

$$\frac{d}{dt}\operatorname{vol}_n(\Sigma_t) = \operatorname{vol}_n(\Sigma_t)$$

under inverse mean curvature flow, whence

$$\frac{d}{dt}\mathcal{H}(\Sigma_t) = \operatorname{vol}_n^{-\frac{n-2}{n}}(\Sigma_t)\left(\frac{d}{dt}\mathcal{F}(\Sigma_t) - \frac{n-2}{n}\mathcal{F}(\Sigma_t)\right). \tag{3.7}$$

Using (4.3s) (with $C = \sqrt{n}$ and $f = 1/H$) we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\Sigma_t) - \frac{n-2}{n}\mathcal{F}(\Sigma_t) = (n-1)\int_{\Sigma_t}|\mathring{A}|^2 - 2(n-1)\int_{\Sigma_t}\operatorname{Hess}\frac{1}{H}:\mathring{A}\\ - 2(n-1)\int_{\Sigma_t}\frac{A^2:\mathring{A}}{H} - \frac{1}{n}\int_{\Sigma_t}(H-\overline{H})^2\\ + \frac{2}{n}\int_{\Sigma_t}(H-\overline{H})\Delta\frac{1}{H} + \frac{2}{n}\int_{\Sigma_t}|A|^2 - \frac{2}{n}\overline{H}\int_{\Sigma_t}\frac{|A|^2}{H}\\ - \frac{(n-1)(n-2)}{n}\int_{\Sigma_t}|\mathring{A}|^2 + \frac{n-2}{n^2}\int_{\Sigma_t}(H-\overline{H})^2\end{aligned}$$

$$
\begin{aligned}
&= (n-1)\left(1-\frac{n-2}{n}\right)\int_{\Sigma_t}|\mathring{A}|^2 - \frac{1}{n}\left(1-\frac{n-2}{n}\right)\int_{\Sigma_t}(H-\overline{H})^2\\
&\quad + \frac{2}{n}\int_{\Sigma_t}|A|^2 - 2(n-1)\int_{\Sigma_t}\frac{A^2{:}\mathring{A}}{H} - \frac{2}{n}\overline{H}\int_{\Sigma_t}\frac{|A|^2}{H}\\
&\quad - 2(n-1)\int_{\Sigma_t}\operatorname{Hess}\frac{1}{H}{:}\mathring{A} + \frac{2}{n}\int_{\Sigma_t}(H-\overline{H})\Delta\frac{1}{H}\\
&= 2\frac{n-1}{n}\int_{\Sigma_t}|A|^2 - 2\frac{n-1}{n^2}\int_{\Sigma_t}H^2 - \frac{2}{n^2}\int_{\Sigma_t}H^2 + \frac{2}{n^2}\overline{H}\int_{\Sigma_t}H\\
&\quad + \frac{2}{n}\int_{\Sigma_t}|A|^2 - 2(n-1)\int_{\Sigma_t}\frac{\operatorname{tr}_g A^3}{H} + 2\frac{n-1}{n}\int_{\Sigma_t}|A|^2\\
&\quad - \frac{2}{n}\overline{H}\int_{\Sigma_t}\frac{|A|^2}{H} - 2(n-1)\int_{\Sigma_t}\operatorname{Hess}\frac{1}{H}{:}A\\
&\quad 2\frac{n-1}{n}\int_{\Sigma_t}H\Delta\frac{1}{H} + \frac{2}{n}\int_{\Sigma_t}H\Delta\frac{1}{H} - \frac{2}{n}\overline{H}\int_{\Sigma_t}\Delta\frac{1}{H}\\
&= -2(n-1)\int_{\Sigma_t}\frac{\operatorname{tr}_g A^3}{H} + 2\frac{2n-1}{n}\int_{\Sigma_t}|A|^2 - 2\frac{1}{n}\int_{\Sigma_t}H^2\\
&\quad - \frac{2}{n}\overline{H}\int_{\Sigma_t}\frac{|\mathring{A}|^2}{H} - 2(n-1)\int_{\Sigma_t}\operatorname{Hess}\frac{1}{H}{:}A + 2\int_{\Sigma_t}H\Delta\frac{1}{H}\\
&\quad - \frac{2}{n}\overline{H}\int_{\Sigma_t}\Delta\frac{1}{H}.
\end{aligned}
$$

Using that Σ_t is closed, we recover, as in the proof of Lemma 3.4, that $\int_{\Sigma_t}\Delta\varphi$ vanishes for any C^2–function φ on Σ_t and that

$$
\begin{aligned}
\int_{\Sigma_t}\operatorname{Hess}\varphi{:}A &= \int_{\Sigma_t}\sum_{i,j=1}^n \nabla_i\nabla^j\varphi A^i{}_j = -\int_{\Sigma_t}\sum_{i,j=1}^n \nabla^j\varphi\nabla_i A^i{}_j\\
&= -\int_{\Sigma_t}\sum_{i,j=1}^n \nabla^j\varphi\nabla_j A^i{}_i = \int_{\Sigma_t}\sum_{i,j=1}^n \nabla_j\nabla^j\varphi A^i{}_i = \int_{\Sigma_t}H\Delta\varphi,
\end{aligned}
$$

where the Codazzi equations have been used. Similarly, we compute

$$
\int_{\Sigma_t}H\Delta\frac{1}{H} = -\int_{\Sigma_t}\sum_{i=1}^n\nabla_i H\nabla_i\frac{1}{H} = \int_{\Sigma_t}\sum_{i=1}^n\nabla_i H\frac{\nabla_i H}{H^2} = \int_{\Sigma_t}\frac{|\nabla H|^2}{H^2},
$$

so that we finally arrive at

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}(\Sigma_t) - \frac{n-2}{n}\mathcal{F}(\Sigma_t) &= -2\left((n-1)\int_{\Sigma_t}\frac{\operatorname{tr}_g A^3}{H} - \frac{2n-1}{n}\int_{\Sigma_t}|A|^2 + \frac{1}{n}\int_{\Sigma_t}H^2\right)\\
&\quad - \frac{2}{n}\overline{H}\int_{\Sigma_t}\frac{|\mathring{A}|^2}{H} - 2(n-2)\int_{\Sigma_t}\frac{|\nabla H|^2}{H^2},
\end{aligned}
$$

which, together with (3.7), immediately implies Lemma 3.7. □

4.3. Proof of Lemma 3.8. Let $\alpha = \min_i \mu_i$ denote the smallest μ_i and define

$$\beta_i = \mu_i - \alpha \qquad (i \in \{1, \dots, n\}),$$

which are all non–negative. Notice that at least one β_i has to vanish. Since $\mu \in \pi$, we have that

$$\sum_{i=1}^n \beta_i = 1 - n\alpha \qquad \text{and} \qquad \alpha \in \left[0, \frac{1}{n}\right].$$

Now write

$$\begin{aligned}
g(\mu) &= (n-1)\sum_{i=1}^n (\alpha+\beta_i)^3 - \frac{2n-1}{n}\sum_{i=1}^n (\alpha+\beta_i)^2 + \frac{1}{n} \\
&= n(n-1)\alpha^3 - (2n-1)\alpha^2 + \frac{1}{n} + (n-1)\sum_{i=1}^n \beta_i^3 \\
&\quad + 3(n-1)\alpha\sum_{i=1}^n \beta_i^2 + 3(n-1)\alpha^2 \sum_{i=1}^n \beta_i \\
&\quad - \frac{2n-1}{n}\sum_{i=1}^n \beta_i^2 - 2\alpha\frac{2n-1}{n}\sum_{i=1}^n \beta_i \\
&= (1-n\alpha)\left(-(n-1)\alpha^2 + \alpha + \frac{1}{n}\right) \\
&\quad + (1-n\alpha)\left(3(n-1)\alpha^2 - 2\alpha\frac{2n-1}{n}\right) \\
&\quad + (n-1)\sum_{i=1}^n \beta_i^3 + \left(3(n-1)\alpha - \frac{2n-1}{n}\right)\sum_{i=1}^n \beta_i^2 \\
&= \frac{1}{n}(1-n\alpha)^2(1-2(n-1)\alpha) \\
&\quad + (n-1)\sum_{i=1}^n \beta_i^3 + \left(3(n-1)\alpha - \frac{2n-1}{n}\right)\sum_{i=1}^n \beta_i^2.
\end{aligned}$$

Using that

$$\sum_{i=1}^n \beta_i^2 = \sum_{i=1}^n \beta_i^{3/2}\beta_i^{1/2} \le \sqrt{\sum_{i=1}^n \beta_i^3}\sqrt{\sum_{i=1}^n \beta_i} = \sqrt{1-n\alpha}\sqrt{\sum_{i=1}^n \beta_i^3},$$

i.e.

$$\sum_{i=1}^n \beta_i^3 \ge \frac{1}{1-n\alpha}\left(\sum_{i=1}^n \beta_i^2\right)^2,$$

yields

$$\begin{aligned} g(\mu) \geq \frac{1}{n}&(1-n\alpha)^2(1-2(n-1)\alpha) \\ &+ \frac{n-1}{1-n\alpha}\left(\sum_{i=1}^n \beta_i^2\right)^2 + \left(3(n-1)\alpha - \frac{2n-1}{n}\right)\sum_{i=1}^n \beta_i^2, \end{aligned}$$

where equality holds only if all β_i vanish, or all non–zero ones are identical.

Consider now the function

$$\phi : x \mapsto \frac{n-1}{1-n\alpha}x^2 + \left(3(n-1)\alpha - \frac{2n-1}{n}\right)x,$$

which is minimal when

$$x = x_{\text{crit}} = (1-n\alpha)\left(\frac{2n-1}{2n(n-1)} - \frac{3}{2}\alpha\right)$$

and monotonically increasing on $[x_{\text{crit}}, +\infty)$. We then have

$$g(\mu) \geq \frac{1}{n}(1-n\alpha)^2(1-2(n-1)\alpha) + \phi\left(\sum_{i=1}^n \beta_i^2\right).$$

Using that at least one β_i vanishes, we can estimate

$$\sum_{i=1}^n \beta_i \leq \sqrt{n-1}\sqrt{\sum_{i=1}^n \beta_i^2},$$

so that

$$\sum_{i=1}^n \beta_i^2 \geq \frac{1}{n-1}\left(\sum_{i=1}^n \beta_i\right)^2 = \frac{(1-n\alpha)^2}{n-1}.$$

Notice that equality holds only if all the β_i vanish, or $(n-1)$ of them are non–zero and identical.

Since

$$\frac{(1-n\alpha)^2}{n-1} \geq x_{\text{crit}} \qquad \forall \alpha \geq -\frac{1}{n(n-3)}$$

and $\alpha \geq 0$ by assumption, we see that

$$\begin{aligned} g(\mu) &\geq \frac{1}{n}(1-n\alpha)^2(1-2(n-1)\alpha) + \phi\left(\frac{(1-n\alpha)^2}{n-1}\right) \\ &= \frac{n-2}{n(n-1)}\alpha(1-n\alpha)^2 \\ &\geq 0, \end{aligned}$$

where the second equality holds only if $\alpha \in \{0, 1/n\}$ and the first, as already mentioned, only if all β_i vanish, or all except one of them are non–zero and equal. In view of the definitions of α and β_i, this leaves precisely the two asserted possibilities and thus concludes the proof. $\square$

CHAPTER 4

About the optimality of some of our results

This chapter is dedicated to producing a few optimality results around the matters discussed so far. We first show that the constant $C = \sqrt{\frac{n}{n-1}}$ on the right–hand side of (3.1) is, in fact, optimal among Ricci–positive (i.e. convex) hypersurfaces. We then argue that the condition $\mathrm{Ric} \geq 0$ is optimal whenever we are in the sub–critical setting (i.e. when $p < n$). Finally, we establish that the constant $C = \sqrt{2}$ is not the appropriate one for *all* two–dimensional surfaces, thereby demonstrating the importance of the additional assumptions in, both, Theorem 3.1 and Theorem 3.3.

Notice that the given counterexamples do not assume unit n–volume. However, this is irrelevant, since both sides of the main estimate scale identically.

Contents

1. The optimality of the constant $C = \sqrt{\frac{n}{n-1}}$ in Theorem 3.1

Proposition 4.1. *Let $n \geq 2$ and $C < \sqrt{\frac{n}{n-1}}$ be given. Then there is a deformation Σ of the standard sphere S^n such that*

$$\left(\int_\Sigma \left| A - \frac{1}{n}\left(\frac{1}{\mathrm{vol}_n(\Sigma)}\int_\Sigma H\right) g \right|^2\right)^{\frac{1}{2}} > C\left(\int_\Sigma \left| A - \frac{H}{n} g\right|^2\right)^{\frac{1}{2}}. \tag{4.1}$$

Moreover, Σ can be chosen arbitrarily close to S^n, ensuring that $\mathrm{Ric}_\Sigma > 0$.

We will prove this proposition by a geometric flow technique.

1.1. Preliminaries. On the set of smooth closed hypersurfaces in $\mathbb{R}^{n+1}$ we introduce the functional

$$\mathcal{F}(\Sigma) = (C^2 - 1) \int_\Sigma |\mathring{A}|^2 - \frac{1}{n} \int_\Sigma \left(H - \frac{1}{\mathrm{vol}_n(\Sigma)} \int_\Sigma H \right)^2 . \tag{4.2}$$

It is immediate to see that $\mathcal{F}(\Sigma)$ is negative if and only if inequality (4.1) holds (compare with (3.2) of the previous chapter). Notice also that $\mathcal{F}(S^n) = 0$.

Let $F_0 : S^n \to \mathbb{R}^{n+1}$ denote the canonical embedding of S^n into $\mathbb{R}^{n+1}$. Given a smooth, real–valued function f on S^n, we consider the family of hypersurfaces Σ_t given by the embeddings $F : S^n \times [0,T] \to \mathbb{R}^{n+1}$ for some $T > 0$ and such that

$$\begin{cases} F(S^n, 0) &= F_0(S^n), \\ \partial_t F(x,t) &= f(x)\nu_t(x) \qquad (x \in S^n), \end{cases}$$

where $\nu_t(x)$ denotes the outer unit normal of Σ_t in $F(x,t)$ (the existence of such an F should follow from, for instance, [**Car65**, §§35–48]).

In what follows, we will calculate the second derivative of $\mathcal{F}(\Sigma_t)$ at $t = 0$ and show that we can choose an $f : S^n \to \mathbb{R}$ such that

$$\begin{cases} \frac{d}{dt}\mathcal{F}(\Sigma_t)\Big|_{t=0} &= 0, \\ \frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)\Big|_{t=0} &< 0. \end{cases}$$

This shows that we can deform S^n slightly to obtain a surface that satisfies (4.1), which proves the optimality of (3.1) (obviously, by continuity, the deformation of S^n thus obtained will have non–negative Ricci curvature for t small enough).

In the sequel, we shall omit the subscript t for the sake of readability.

1.2. The first variation of $\mathcal{F}$. We choose any coordinate patch and compute

$$\partial_t g_{ij} = \partial_t \langle \partial_i F, \partial_j F \rangle = \langle \partial_i \partial_t F, \partial_j F \rangle + \langle \partial_i F, \partial_j \partial_t F \rangle = 2fA_{ij}, \tag{4.3a}$$

$$\partial_t g^{ij} = -\sum_{k,l=1}^n g^{ik} g^{jl} \partial_t g_{kl} \overset{(4.3a)}{=} -2fA^{ij}, \tag{4.3b}$$

$$\begin{aligned} \partial_t \sqrt{\det g} &= \frac{1}{2\sqrt{\det g}} \partial_t \det g = \frac{1}{2\sqrt{\det g}} \det g \operatorname{tr}_g (\partial_t g) \overset{(4.3a)}{=} f \operatorname{tr}_g(A) \sqrt{\det g} \\ &= fH\sqrt{\det g}. \end{aligned} \tag{4.3c}$$

On the other hand,

$$\partial_t \partial_i F = (\partial_i f)\nu + f(\partial_i \nu) = (\partial_i f)\nu + f \sum_{k=1}^n A^k{}_i \partial_k F, \tag{4.3d}$$

$$\partial_t \nu = \sum_{i,j=1}^n \langle \partial_t \nu, \partial_i F \rangle g^{ij} \partial_j F = -\sum_{i,j=1}^n \langle \nu, \partial_t \partial_i F \rangle g^{ij} \partial_j F \overset{(4.3d)}{=} -\nabla f. \tag{4.3e}$$

Consequently,

$$\begin{aligned}\partial_t A_{ij} &= \partial_t \langle \partial_i \nu, \partial_j F\rangle = \langle \partial_i \partial_t \nu, \partial_j F\rangle + \langle \partial_i \nu, \partial_t \partial_j F\rangle \\ &\overset{(4.3d)\&(4.3e)}{=} - \langle \partial_i \nabla f, \partial_j F\rangle + f \sum_{k=1}^n A^k{}_j A_{ki} = -\operatorname{Hess}_{ij} f + f(A^2)_{ij}\end{aligned}\tag{4.3f}$$

and

$$\partial_t H = \sum_{i,j=1}^n (\partial_t g^{ij}) A_{ij} + \sum_{i,j=1}^n g^{ij}(\partial_t A_{ij}) \overset{(4.3b)\&(4.3f)}{=} -\Delta f - f\,|A|^2\,,\tag{4.3g}$$

from which

$$\begin{aligned}\partial_t \mathring{A}_{ij} &= \partial_t A_{ij} - \frac{1}{n}(\partial_t H) g_{ij} - \frac{1}{n} H(\partial_t g_{ij}) \\ &\overset{(4.3a),(4.3f)\&(4.3g)}{=} -\operatorname{Hess}_{ij} f + \frac{1}{n}\Delta f g_{ij} + f\left((A^2)_{ij} - \frac{1}{n} H A_{ij}\right) \\ &\quad + \frac{1}{n} f\left(|A|^2 g_{ij} - H A_{ij}\right) \\ &= -\operatorname{Hess}_{ij} f + \frac{1}{n}\Delta f g_{ij} + f(A\mathring{A})_{ij} \\ &\quad + \frac{1}{n} f\left(\left(|\mathring{A}|^2 + \frac{H^2}{n}\right) g_{ij} - H\left(\mathring{A}_{ij} + \frac{1}{n} H g_{ij}\right)\right), \\ &= -\operatorname{Hess}_{ij} f + \frac{1}{n}\Delta f g_{ij} + f(A\mathring{A})_{ij} + \frac{1}{n} f\,|\mathring{A}|^2 g_{ij} - \frac{1}{n} f H \mathring{A}_{ij},\end{aligned}\tag{4.3h}$$

$$\begin{aligned}\partial_t |A|^2 &= 2 \sum_{i,j,k,l=1}^n g^{ik} A_{kl}\left((\partial_t g^{jl}) A_{ij} + g^{jl}(\partial_t A_{ij})\right) \\ &\overset{(4.3b)\&(4.3f)}{=} -4f \sum_{i,j,l=1}^n A^i{}_l A^{lj} A_{ji} - 2\operatorname{Hess} f{:}A + 2 \sum_{i,j,k=1}^n f A^{ij} A_j{}^k A_{ki} \\ &= -2\operatorname{Hess} f{:}A - 2f \operatorname{tr}_g(A^3)\end{aligned}\tag{4.3i}$$

and

$$\begin{aligned}\partial_t |\mathring{A}|^2 &= \partial_t\left(|A|^2 - \frac{1}{n} H^2\right) = \partial_t |A|^2 - \frac{2}{n} H(\partial_t H) \\ &\overset{(4.3g)\&(4.3i)}{=} -2\operatorname{Hess} f{:}A - 2f\operatorname{tr}_g(A^3) + \frac{2}{n} H\Delta f + \frac{2}{n} f H\,|A|^2 \\ &= -2\operatorname{Hess} f{:}\mathring{A} - 2f A^2{:}\mathring{A}.\end{aligned}\tag{4.3j}$$

By (4.3c), we have for any smooth function $\varphi : S^n \times [0,T] \to \mathbb{R}$

$$\frac{d}{dt}\int_\Sigma \varphi \overset{(4.3c)}{=} \int_\Sigma f H \varphi + \int_\Sigma \partial_t \varphi.\tag{4.3k}$$

As a consequence, we get

$$\frac{d}{dt}\int_\Sigma |\mathring{A}|^2 \overset{(4.3j)\&(4.3k)}{=} \int_\Sigma fH\,|\mathring{A}|^2 - 2\int_\Sigma \operatorname{Hess} f{:}\mathring{A} - 2\int_\Sigma fA^2{:}\mathring{A}. \tag{4.3l}$$

On the other hand, if we introduce the notation

$$\overline{\varphi} = \frac{1}{\operatorname{vol}_n(\Sigma)}\int_\Sigma \varphi,$$

we have

$$\frac{d}{dt}\overline{\varphi} \overset{(4.3k)\&(4.3n)}{=} \frac{1}{\operatorname{vol}_n(\Sigma)}\left(\int_\Sigma fH(\varphi-\overline{\varphi}) + \int_\Sigma \partial_t\varphi\right), \tag{4.3m}$$

since

$$\frac{d}{dt}\frac{1}{\operatorname{vol}_n(\Sigma)} = -\frac{1}{\operatorname{vol}_n^2(\Sigma)}\frac{d}{dt}\int_\Sigma 1 \overset{(4.3k)}{=} -\frac{1}{\operatorname{vol}_n^2(\Sigma)}\int_\Sigma fH. \tag{4.3n}$$

Therefore,

$$\partial_t(\varphi-\overline{\varphi}) \overset{(4.3m)}{=} \partial_t\varphi - \overline{(\partial_t\varphi)} - \overline{(fH(\varphi-\overline{\varphi}))} \tag{4.3o}$$

and

$$\begin{aligned}\frac{d}{dt}\int_\Sigma (\varphi-\overline{\varphi})^2 &\overset{(4.3k)\&(4.3o)}{=} \int_\Sigma fH(\varphi-\overline{\varphi})^2 + 2\int_\Sigma (\varphi-\overline{\varphi})\left(\partial_t(\varphi-\overline{\varphi})\right)\\ &= \int_\Sigma fH(\varphi-\overline{\varphi})^2 + 2\int_\Sigma (\varphi-\overline{\varphi})\partial_t\varphi,\end{aligned} \tag{4.3p}$$

where we have used that $\int_\Sigma(\varphi-\overline{\varphi})=0$. It follows that

$$\partial_t(H-\overline{H}) \overset{(4.3g)\&(4.3o)}{=} -\Delta f - f\,|A|^2 + \overline{(\Delta f + f\,|A|^2)} - \overline{(fH(H-\overline{H}))} \tag{4.3q}$$

and

$$\frac{d}{dt}\int_\Sigma (H-\overline{H})^2 \overset{(4.3g)\&(4.3p)}{=} \int_\Sigma fH(H-\overline{H})^2 - 2\int_\Sigma (H-\overline{H})\Delta f - 2\int_\Sigma f\,|A|^2\,(H-\overline{H}). \tag{4.3r}$$

Putting all this together finally yields

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\Sigma) \overset{(4.3l)\&(4.3r)}{=} &(C^2-1)\left(\int_\Sigma fH\,|\mathring{A}|^2 - 2\int_\Sigma \operatorname{Hess} f{:}\mathring{A} - 2\int_\Sigma fA^2{:}\mathring{A}\right)\\ &-\frac{1}{n}\left(\int_\Sigma fH(H-\overline{H})^2 - 2\int_\Sigma (H-\overline{H})\Delta f - 2\int_\Sigma f\,|A|^2\,(H-\overline{H})\right).\end{aligned} \tag{4.3s}$$

Since $\mathring{A}$ vanishes on $S^n = \Sigma|_{t=0}$, whereas H is constant there, we immediately see that, for *any* f,

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{F}(\Sigma) = 0.$$

Consequently, S^n is a critical point of $\mathcal{F}$ and it makes sense to study the second order variation of $\mathcal{F}$ at $t = 0$. This is precisely what we wish to do next.

1.3. The second variation of $\mathcal{F}$. If we denote by η the canonical metric on S^n, we have at $t = 0$

$$\begin{aligned} g|_{t=0} &= \eta, \\ A|_{t=0} &= \eta, \\ H|_{t=0} &= n, \\ \mathring{A}|_{t=0} &= 0 \end{aligned}$$

and

$$(H - \overline{H})|_{t=0} = 0.$$

Omitting any subscript S^n on quantities or operators which are now evaluated on $S^n = \Sigma_t\big|_{t=0}$ rather than on Σ_t, we obtain from (4.3h), (4.3j) and (4.3q)

$$\partial_t\big|_{t=0} \mathring{A} \overset{(4.3h)}{=} -\operatorname{Hess} f + \frac{1}{n}(\Delta f)\eta, \tag{4.4a}$$

$$\partial_t\big|_{t=0} |\mathring{A}|^2 \overset{(4.3j)}{=} 0, \tag{4.4b}$$

$$\partial_t\big|_{t=0} (H - \overline{H}) \overset{(4.3q)}{=} -\Delta f - f|\eta|^2 + \overline{(\Delta f + f|\eta|^2)} \tag{4.4c}$$

and

$$\partial_t\big|_{t=0} (H - \overline{H})^2 = 2\left((H - \overline{H})(\partial_t(H - \overline{H}))\right)\Big|_{t=0} = 0. \tag{4.4d}$$

In view of the vanishing of $\mathring{A}$ and $(H - \overline{H})$ at $t = 0$, these are sufficient to compute the second variation of $\mathcal{F}$. Indeed, from (4.3s) we obtain using the identities above

(4.4e)

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{F}(\Sigma) = (C^2 - 1)\Bigg(&-2\int_{S^n} \operatorname{Hess} f \colon \Big(-\operatorname{Hess} f + \frac{1}{n}\eta\Delta f\Big) \\ &- 2\int_{S^n} f\eta^2 \colon \Big(-\operatorname{Hess} f + \frac{1}{n}\eta\Delta f\Big)\Bigg) \\ - \frac{1}{n}\Bigg(&-2\int_{S^n} \Delta f\left(-\Delta f - fn^2 + \overline{(\Delta f + fn^2)}\right) \\ &- 2\int_{S^n} fn^2\left(-\Delta f - fn^2 + \overline{(\Delta f + fn^2)}\right)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= (C^2-1)\left(2\int_{S^n} \operatorname{Hess} f : \overset{\circ}{\operatorname{Hess}} f + 2\int_{S^n} f\Delta f - 2\int_{S^n} \frac{n}{n} f\Delta f\right) \\
&\quad - \frac{2}{n}\left(\int_{S^n} \Delta f\left(\Delta f + nf - \overline{(\Delta f + nf)}\right)\right. \\
&\qquad\qquad \left. + \int_{S^n} nf\left(\Delta f + nf - \overline{(\Delta f + nf)}\right)\right) \\
&= 2(C^2-1)\int_{S^n} |\overset{\circ}{\operatorname{Hess}} f|^2 - \frac{2}{n}\int_{S^n}\left(\Delta f + nf - \overline{(\Delta f + nf)}\right)^2,
\end{aligned}
$$

where $\overset{\circ}{\operatorname{Hess}} f = \operatorname{Hess} f - \frac{1}{n}(\Delta f)g$ is the traceless part of $\operatorname{Hess} f$, and we have used that $\int_{S^n}(\varphi - \overline{\varphi}) = 0$ for any smooth function φ on S^n.

1.4. Proof of Proposition 4.1. By the same calculation as in (3.3) of Section 1, Chapter 3, we have

$$
\int_{S^n} |\operatorname{Hess} f|^2 = \int_{S^n} (\Delta f)^2 - \int_{S^n} \operatorname{Ric}(\nabla f, \nabla f) = \int_{S^n} (\Delta f)^2 - (n-1)\int_{S^n} |\nabla f|^2,
$$

since $\operatorname{Ric} = (n-1)\eta$ on S^n. Consequently,

$$
\int_{S^n} |\overset{\circ}{\operatorname{Hess}} f|^2 = \frac{n-1}{n}\int_{S^n} (\Delta f)^2 - (n-1)\int_{S^n} |\nabla f|^2 .
$$

On the other hand, if we assume that $\overline{f} = 0$, we obtain through partial integration

$$
\begin{aligned}
\int_{S^n} \left(\Delta f + nf - \overline{(\Delta f + nf)}\right)^2 &= \int_{S^n} (\Delta f)^2 + 2n\int_{S^n} f\Delta f + n^2\int_{S^n} f^2 \\
&\quad - \frac{1}{\operatorname{vol}_n(S^n)}\left(\int_{S^n} \Delta f + n\int_{S^n} f\right)^2 \\
&= \int_{S^n} (\Delta f)^2 - 2n\int_{S^n} |\nabla f|^2 + n^2\int_{S^n} f^2.
\end{aligned}
$$

Hence, (4.4e) is equivalent to

(4.5)

$$
\begin{aligned}
\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{F}(\Sigma) &= 2\left(C^2\frac{n-1}{n} - 1\right)\int_{S^n} (\Delta f)^2 - 2n\left(C^2\frac{n-1}{n} - 1\right)\int_{S^n} |\nabla f|^2 \\
&\quad + 2\left(\int_{S^n} |\nabla f|^2 - n\int_{S^n} f^2\right) \\
&= 2\left(C^2\frac{n-1}{n} - 1\right)\int_{S^n} \left((\Delta f)^2 - n\,|\nabla f|^2\right) + 2\int_{S^n} \left(|\nabla f|^2 - nf^2\right),
\end{aligned}
$$

as long as we require $\int_{S^n} f = 0$.

Now let f be a spherical harmonic of order k, where $k > 1$ will be chosen below, and set $\alpha = \left(C^2 \frac{n-1}{n} - 1\right)$. f satisfies $-\Delta f = k(k+n-1)f$, $\overline{f} = 0$, and (4.5) becomes

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{F}(\Sigma) &= 2\alpha \left(k^2(k+n-1)^2 - nk(k+n-1)\right) \int_{S^n} f^2 \\ &\quad + 2\left(k(k+n-1) - n\right) \int_{S^n} f^2 \\ &= 2\left(\int_{S^n} f^2\right)\Big(\alpha k(k+n-1)\big(k(k+n-1)-n\big) \\ &\qquad\qquad + \big(k(k+n-1)-n\big)\Big) \\ &= 2(k-1)(k+n)\left(\int_{S^n} f^2\right)\Big(\alpha k(k+n-1)+1\Big). \end{aligned} \tag{4.6}$$

For $k > 1$, this quantity is negative whenever

$$\alpha k(k+n-1) + 1 < 0,$$

which can be achieved as soon as

$$k > \sqrt{\frac{(n-1)^2}{4} + \frac{n}{n - C^2(n-1)}} - \frac{n-1}{2},$$

since we had assumed that $\alpha = \left(C^2 \frac{n-1}{n} - 1\right) < 0$. As a result, there exists a function on S^n such that $\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F}(\Sigma) < 0$ and the Proposition is proved. $\square$

2. The optimality of the assumption Ric ≥ 0 for the general sub–critical estimate

In this section we prove that, for $p \in [1, n)$, there are surfaces for which estimate (3.1) fails (for *any* constant) if we don't assume the Ricci curvature to be non–negative. This will be a direct consequence of the following proposition which is an easy generalisation of Proposition 7.1 in [**DLM05**].

Proposition 4.2 (De Lellis, Müller, P.). *Let $n \geq 2$ be given. There exists a family of smooth, closed, connected hypersurfaces $\Sigma_\epsilon \subset \mathbb{R}^{n+1}$ such that:*

$$C \geq \mathrm{vol}_n(\Sigma_\epsilon) \geq c > 0, \qquad \textit{for every } \epsilon > 0; \tag{4.7a}$$

$$\lim_{\epsilon \searrow 0} \mathrm{vol}_n(\{q \in \Sigma_\epsilon \,|\, \mathrm{Ric}(q) < 0\}) = 0; \tag{4.7b}$$

$$\lim_{\epsilon \searrow 0} \int_{\Sigma_\epsilon} |\mathring{A}|^p = 0, \qquad \textit{for every } p \in [1, n); \tag{4.7c}$$

(4.7d) Σ_ϵ *converges in the Hausdorff topology to the union of two round spheres;*

and

$$\lim_{\epsilon\searrow 0}\left(\inf_{\lambda}\int_{\Sigma_\epsilon}|A-\lambda g|^p\right)>0, \qquad \textit{for every } p\in[1,n). \tag{4.7e}$$

In particular, this immediately implies

Corollary 4.3. *Assume $n>2$. Then, for every $C>0$ and every $\delta>0$ we can find a smooth, closed hypersurface Σ in $\mathbb{R}^{n+1}$ such that*

$$\left(\int_\Sigma\left|A-\frac{\overline{H}}{n}g\right|^2\right)^{\frac{1}{2}}>C\left(\int_\Sigma|\mathring{A}|^2\right)^{\frac{1}{2}},$$

and where the portion of Σ on which the Ricci curvature is negative has n–dimensional volume smaller than δ.

The proof of Proposition 4.2 presented below follows closely the construction in [**DLM05**]. The idea is to consider two round spheres of radii 1 and 1/2, respectively, and glue them together with a small hyperbolic neck so that the L^p–norm of the second fundamental form on that neck becomes arbitrarily small. As in [**DLM05**], we choose a catenoidal neck to simplify the computations.

2.1. Preliminaries. Let I be a closed interval. We call a hypersurface $\Sigma\subset\mathbb{R}^{n+1}$ *hypersurface of revolution* (around the x^{n+1}–axis) if there exist two smooth functions f and h on I, with $f>0$ and $(f')^2+(h')^2>0$ on the interior $\text{int}I$ of I, and such that Σ is parametrised by the map

$$\begin{aligned} F\,:\,S^{n-1}\times I &\to \mathbb{R}^{n+1}\\ (x,t) &\mapsto (f(t)\Phi(x),h(t))\,,\end{aligned}$$

where Φ denotes the canonical embedding of S^{n-1} into $\mathbb{R}^n$. We call the curve $\phi: I\to\mathbb{R}^2, t\mapsto(f(t),h(t))$ the *generating curve* of Σ, and assume it is injective.

In the coordinates (x,t), the metric of Σ and its inverse are given by

$$g=\begin{pmatrix} f^2\eta & 0\\ 0 & (f')^2+(h')^2\end{pmatrix}\quad\text{and}\quad g^{-1}=\begin{pmatrix} f^{-2}\eta^{-1} & 0\\ 0 & \frac{1}{(f')^2+(h')^2}\end{pmatrix},$$

where η denotes the canonical metric on S^{n-1}. Consequently,

$$\sqrt{\det g}=f^{n-1}\sqrt{(f')^2+(h')^2}\sqrt{\det\eta}. \tag{4.8a}$$

One easily checks that the outward unit normal on Σ is given by

$$\nu=\frac{1}{\sqrt{(f')^2+(h')^2}}\left(h'\Phi,-f'\right).$$

Then one computes immediately

$$A=\frac{1}{\sqrt{(f')^2+(h')^2}}\begin{pmatrix} fh'\eta & 0\\ 0 & f'h''-f''h'\end{pmatrix} \tag{4.8b}$$

and

$$H = \frac{(n-1)\frac{h'}{f} + \frac{f'h'' - f''h'}{(f')^2 + (h')^2}}{\sqrt{(f')^2 + (h')^2}}. \tag{4.8c}$$

2.2. Detailed construction. For $\epsilon \in \left(0, 2^{-(n-1)}\right]$, consider the three families S^u_ϵ, C_ϵ and S^l_ϵ of hypersurfaces of revolution given by the parametrisations

$$F_{S^u_\epsilon} : S^{n-1} \times \left[z^u_\epsilon, c^u_\epsilon + \frac{1}{2}\right], \qquad (x,t) \mapsto \left(\sqrt{\frac{1}{4} - (c^u_\epsilon - t)^2}\Phi(x), t\right),$$

$$F_{C_\epsilon} : S^{n-1} \times \left[t^l_\epsilon, t^u_\epsilon\right],$$

$$(x,t) \mapsto \left(\sqrt[n-1]{\epsilon \cosh\left(\frac{(n-1)t}{\epsilon}\right)}\Phi(x), \int_0^t \left(\epsilon \cosh\left(\frac{(n-1)s}{\epsilon}\right)\right)^{-\frac{n-2}{n-1}} ds\right),$$

and

$$F_{S^l_\epsilon} : S^{n-1} \times \left[c^l_\epsilon - 1, z^l_\epsilon\right], \qquad (x,t) \mapsto \left(\sqrt{1 - (c^l_\epsilon - t)^2}\Phi(x), t\right),$$

respectively, where

$$t^u_\epsilon = \frac{\epsilon}{n-1} \operatorname{arccosh}\left(\frac{\sqrt[n]{2}}{2\sqrt[n]{\epsilon}}\right), \qquad t^l_\epsilon = -\frac{\epsilon}{n-1} \operatorname{arccosh}\left(\frac{1}{\sqrt[n]{\epsilon}}\right),$$

$$z^u_\epsilon = \int_0^{t^u_\epsilon} \left(\epsilon \cosh\left(\frac{(n-1)s}{\epsilon}\right)\right)^{-\frac{n-2}{n-1}} ds, \quad z^l_\epsilon = \int_0^{t^l_\epsilon} \left(\epsilon \cosh\left(\frac{(n-1)s}{\epsilon}\right)\right)^{-\frac{n-2}{n-1}} ds$$

and

$$c^u_\epsilon = z^u_\epsilon + \sqrt{\frac{1}{4} - \frac{\epsilon^{2/n}}{2^{2/n}}}, \qquad c^l_\epsilon = z^l_\epsilon - \sqrt{1 - \epsilon^{2/n}}.$$

The parameters $c^{u/l}_\epsilon$, $z^{u/l}_\epsilon$ and $t^{u/l}_\epsilon$ were chosen such that, for each $\epsilon \in \left(0, 2^{-(n-1)}\right]$, $\Sigma_\epsilon = S^u_\epsilon \cup C_\epsilon \cup S^l_\epsilon$ is a closed hypersurface of revolution, generated by a C^1–curve which is piecewise C^∞. Its constituents are a portion S^u_ϵ of a sphere of radius 1/2 and a portion S^l_ϵ of a sphere of radius 1 (so that $\mathring{A}\big|_{S^{u/l}_\epsilon} = 0$), connected by a catenoidal neck C_ϵ (so that $H\big|_{C_\epsilon} = 0$, as is immediately verified using (4.8c)). The sets $\gamma^u_\epsilon = S^u_\epsilon \cap C_\epsilon$ and $\gamma^l_\epsilon = S^l_\epsilon \cap C_\epsilon$ on which the constituents touch are $(n-1)$–dimensional spheres of radius $\sqrt[n]{\epsilon}$ and $\sqrt[n]{\epsilon/2}$, respectively (see Figure 4.1).

Remark 4.4. *It might not be completely obvious why the resulting surface should be C^1. At the top and bottom it is clear that only a coordinate singularity occurs. At the two junctions $\gamma^{u/l}_\epsilon$, however, a short computation shows that the tangent spaces on both sides coincide. Hence we could re–parametrise the three generating curves to get a single C^1–curve (for instance as the graph over $[c^l_\epsilon - 1, c^u_\epsilon + 1/2]$ in the variable x^{n+1}).*

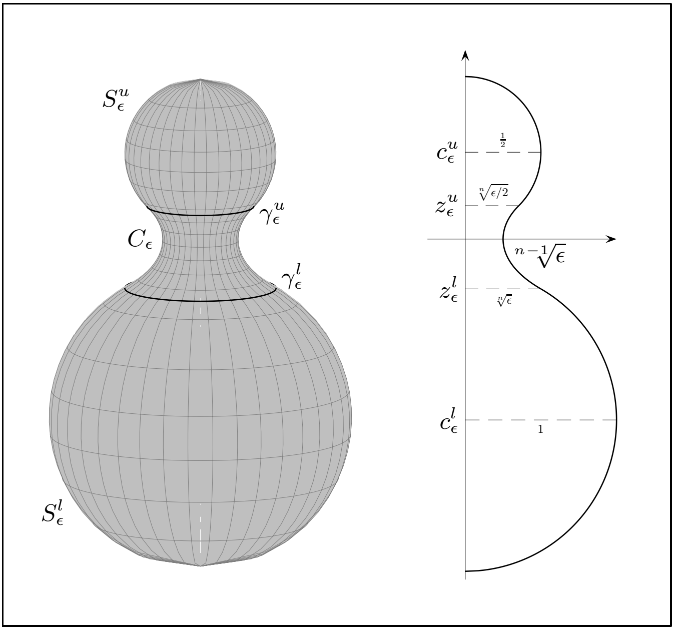

FIGURE 4.1.

2.3. Proof of Proposition 4.2. We use the construction of the previous subsection. One immediately sees that, as $\epsilon \searrow 0$,

$$z^u_\epsilon \searrow 0\,, \quad z^l_\epsilon \nearrow 0\,, \quad c^u_\epsilon \searrow \frac{1}{2} \quad \text{and} \quad c^l_\epsilon \nearrow -1.$$

Therefore, since the radii of γ^u_ϵ and γ^l_ϵ also converge to zero, we conclude that

(4.9) S^u_ϵ and S^l_ϵ converge, respectively, to a sphere S^u_0 of radius $\dfrac{1}{2}$ and to a sphere S^l_0 of radius 1, which are tangent at the origin in $\mathbb{R}^{n+1}$.

Now observe that

(4.10a)
$$\cosh\left(\frac{(n-1)t}{\epsilon}\right) \in \left[1, \max\left\{2^{-\frac{n-1}{n}}\epsilon^{-\frac{1}{n}}, \epsilon^{-\frac{1}{n}}\right\}\right] = \left[1, \epsilon^{-\frac{1}{n}}\right] \qquad \text{on } \left[t^l_\epsilon, t^u_\epsilon\right].$$

Moreover,

$$t^u_\epsilon - t^l_\epsilon = \frac{\epsilon}{n-1}\left(\operatorname{arccosh}\left(2^{-\frac{n-1}{n}}\epsilon^{-\frac{1}{n}}\right) + \operatorname{arccosh}\left(\epsilon^{-\frac{1}{n}}\right)\right)$$

(4.10b)
$$\in \left[\frac{\epsilon}{n(n-1)}\ln\left(2^{-(n-1)}\epsilon^{-2}\right), \frac{\epsilon}{n(n-1)}\ln\left(2^{n+1}\epsilon^{-2}\right)\right],$$

which can be seen using the trivial estimate

$$\ln t \leq \operatorname{arccosh} t = \ln\left(t + \sqrt{t^2 - 1}\right) \leq \ln(2t) \qquad \forall t \geq 1.$$

As a consequence, setting $\omega_{n-1} = \mathrm{vol}_{n-1}(S^{n-1})$,

$$\mathrm{vol}_n(C_\epsilon) = \int_{C_\epsilon} 1 = \omega_{n-1} \int_{t_\epsilon^l}^{t_\epsilon^u} \sqrt{\det g}\, dt \overset{(4.8a)}{=} \omega_{n-1} \int_{t_\epsilon^l}^{t_\epsilon^u} \epsilon^{\frac{1}{n-1}} \cosh^{\frac{n}{n-1}}\left(\frac{(n-1)t}{\epsilon}\right) dt$$

(4.11)

$$\overset{(4.10a)\&(4.10b)}{\leq} \frac{\omega_{n-1}}{n(n-1)} \epsilon \ln\left(2^{n+1}\epsilon^{-2}\right) \overset{\epsilon \searrow 0}{\longrightarrow} 0,$$

which implies (4.7b) and, together with (4.9), (4.7d). Also, we immediately get (4.7a). Thus it remains to show (4.7c) and (4.7e).

For (4.7c), we first use (4.8b) to obtain

$$|A|^p\Big|_{C_\epsilon} = n^{\frac{p}{2}}(n-1)^{\frac{p}{2}} \epsilon^{-\frac{p}{n-1}} \cosh^{-\frac{np}{n-1}}\left(\frac{(n-1)t}{\epsilon}\right).$$

Then, with (4.8a), we calculate

$$\begin{aligned}\int_{C_\epsilon} |A|^p &= \omega_{n-1} \int_{t_\epsilon^l}^{t_\epsilon^u} |A|^p \sqrt{\det g}\, dt \\ &= \omega_{n-1} \int_{t_\epsilon^l}^{t_\epsilon^u} n^{\frac{p}{2}}(n-1)^{\frac{p}{2}} \epsilon^{-\frac{p-1}{n-1}} \cosh^{-\frac{n(p-1)}{n-1}}\left(\frac{(n-1)t}{\epsilon}\right) dt.\end{aligned}$$

Finally, we use (4.10a) and (4.10b) to conclude

$$\int_{C_\epsilon} |A|^p \in \left[(n(n-1))^{\frac{p-2}{2}} \epsilon \ln\left(2^{-(n-1)}\epsilon^{-2}\right), (n(n-1))^{\frac{p-2}{2}} \epsilon^{\frac{n-p}{n-1}} \ln\left(2^{-(n-1)}\epsilon^{-2}\right)\right].$$

Hence

$$\lim_{\epsilon \searrow 0} \int_{C_\epsilon} |A|^p = 0, \tag{4.12}$$

and the fact that H vanishes on C_ϵ, whereas $\mathring{A}$ vanishes on S_ϵ^l and S_ϵ^u, implies immediately (4.7c).

For (4.7e), we use (4.9), (4.11) and (4.12) to get

$$\begin{aligned}\lim_{\epsilon \searrow 0}\left(\inf_\lambda \int_{\Sigma_\epsilon} |A - \lambda g|^p\right) &= \inf_\lambda \left(\int_{S_0^l} |A - \lambda g|^p + \int_{S_0^u} |A - \lambda g|^p\right) \\ &= \inf_\lambda \left(\omega_n n^{\frac{p}{2}} |1 - \lambda|^p + \frac{\omega_n n^{\frac{p}{2}}}{2^n}\left|\frac{1}{2} - \lambda\right|^p\right) > 0.\end{aligned}$$

As already mentioned in the previous subsection, the hypersurfaces Σ_ϵ are only C^1. They are, however, hypersurfaces of revolution. The curves generating them are C^1 and piecewise C^∞ (see Remark 4.4), bearing two jump discontinuities in their higher derivatives. A standard smoothing argument therefore yields a family of hypersurfaces of revolution satisfying the requirements of Proposition 4.2. This finishes the proof. □

Remark 4.5. *The interested reader might wonder whether the surfaces just constructed fulfil (in the case $n = 2$) the assumptions of Theorem 3.3. In view of our comments at the beginning of Section 3 in Chapter 3, we only need to check whether these surfaces are star–shaped. However, an easy calculation shows that they are not, as soon as*

$$\left(\sqrt{1-\frac{\epsilon}{r}}\operatorname{arccosh}\sqrt{\frac{r}{\epsilon}}\right)\Bigg|_{r=\frac{1}{2}} > 1.$$

But this is always true for ϵ small enough.

3. Generic two–dimensional surfaces fail to satisfy the L^2–estimate with $C = \sqrt{2}$

In this section we want to show that there are surfaces in $\mathbb{R}^3$ for which estimate (3.4) fails, thereby demonstrating that additional assumptions, as in theorems 3.1 and 3.3, are essential. More precisely, we wish to prove the following proposition.

Proposition 4.6 (De Lellis, Topping, P.). *There exists a family of smooth, closed, connected surfaces $\Sigma_\epsilon \subset \mathbb{R}^3$ such that*

$$C \geq \mathrm{vol}_n(\Sigma_\epsilon) \geq c > 0, \qquad \text{for every } \epsilon > 0; \tag{4.13a}$$

$$\Sigma_\epsilon \text{ converges in the Hausdorff topology to a double copy of a round sphere;} \tag{4.13b}$$

and

$$\lim_{\epsilon \searrow 0} \frac{\displaystyle\int_{\Sigma_\epsilon} \left| A - \frac{1}{2\mathrm{vol}_2(\Sigma_\epsilon)} \left(\int_{\Sigma_\epsilon} H \right) g \right|^2}{\displaystyle\int_{\Sigma_\epsilon} |\mathring{A}|^2} = 3. \tag{4.13c}$$

The idea is very similar to the one in Section 2, the only difference being that, this time, we attach two concentric spheres of almost the same radius by a catenoidal neck. Of course, to do this smoothly enough (i.e. at least C^1), there will be a transition zone to take into account. Below we give the construction in detail.

3.1. Detailed construction.

Let $r > 0$. We use the notations and calculations of Section 2. For $\epsilon \in (0, r)$, consider the four families S^o_ϵ, P_ϵ, C_ϵ and S^i_ϵ of surfaces of revolution given by the parametrisations

$$F_{S_\epsilon^o} : S^1 \times [c_\epsilon - r - \delta_\epsilon, z_\epsilon], \qquad (x,t) \mapsto \left(\sqrt{(r+\delta_\epsilon)^2 - (c_\epsilon - t)^2} \Phi(x), t \right),$$

$$F_{P_\epsilon} : S^1 \times \left[-\rho_\epsilon - \sqrt{\frac{\epsilon}{r}} \delta_\epsilon, -\rho_\epsilon \right],$$

$$(x,t) \mapsto \left(-t\Phi(x), z_\epsilon + \alpha_\epsilon \left(\left(\sqrt{\frac{\epsilon}{r}} \frac{\delta_\epsilon}{2} \right)^2 - \left(t + \rho_\epsilon + \sqrt{\frac{\epsilon}{r}} \frac{\delta_\epsilon}{2} \right)^2 \right) \right),$$

$$F_{C_\epsilon} : S^1 \times [-z_\epsilon, z_\epsilon], \qquad (x,t) \mapsto \left(\epsilon \cosh\left(\frac{t}{\epsilon} \right) \Phi(x), -t \right),$$

and

$$F_{S_\epsilon^i} : S^1 \times [z_\epsilon, r - c_\epsilon], \qquad (x,t) \mapsto \left(\sqrt{r^2 - (c_\epsilon + t)^2} \Phi(x), -t \right),$$

respectively, where the parameters

$$z_\epsilon = \epsilon \operatorname{arccosh}\left(\sqrt{\frac{r}{\epsilon}} \right), \qquad c_\epsilon = -r\sqrt{1 - \frac{\epsilon}{r}} - z_\epsilon,$$

$$\delta_\epsilon = \frac{2z_\epsilon}{\sqrt{1 - \frac{\epsilon}{r}}}, \qquad \alpha_\epsilon = \frac{1}{2z_\epsilon}$$

and

$$\rho_\epsilon = \sqrt{r\epsilon}$$

were chosen such that, for each $\epsilon \in (0, r)$, $\Sigma_\epsilon = S_\epsilon^i \cup C_\epsilon \cup P_\epsilon \cup S_\epsilon^o$ is a closed surface of revolution, generated by a C^1–curve which is piecewise C^∞. Its constituents are a portion S_ϵ^i of a sphere of radius r inside a portion S_ϵ^o of a concentric sphere of radius $r + \delta_\epsilon$ (so that $\mathring{A}\big|_{S_\epsilon^{i/o}} = 0$), connected by a catenoidal neck C_ϵ (so that $H\big|_{C_\epsilon} = 0$) and a transitional region P_ϵ the cross–section of which is a piece of parabola. The sets $\gamma_\epsilon^i = S_\epsilon^i \cap C_\epsilon$, $\gamma_\epsilon^m = C_\epsilon \cap P_\epsilon$ and $\gamma_\epsilon^o = S_\epsilon^o \cap P_\epsilon$ on which the constituents touch are circles of radius ρ_ϵ, ρ_ϵ and $\rho_\epsilon + \sqrt{\frac{\epsilon}{r}}$, respectively (see Figure 4.2). Notice that a remark analogous to Remark 4.4 holds here also.

3.2. Proof of Proposition 4.6. In the construction of the previous subsection, letting $\epsilon \searrow 0$, one immediately sees that

$$z_\epsilon \searrow 0, \qquad c_\epsilon \nearrow -r \qquad \text{and} \qquad \delta_\epsilon \searrow 0.$$

Since the radii of γ_ϵ^i, γ_ϵ^m and γ_ϵ^o also converge to zero, we conclude that

(4.14)
S_ϵ^i and S_ϵ^o converge each to a sphere S_0 of radius r, with opposite orientations.

We now prove that the areas of P_ϵ and C_ϵ converge to zero as $\epsilon \searrow 0$. Let $\tau_\epsilon(t)$ denote the derivative with respect to t of the second component of F_{P_ϵ}, i.e.

$$\tau_\epsilon(t) = -\frac{t + \rho_\epsilon}{z_\epsilon} - \frac{1}{\sqrt{\frac{r}{\epsilon} - 1}}.$$

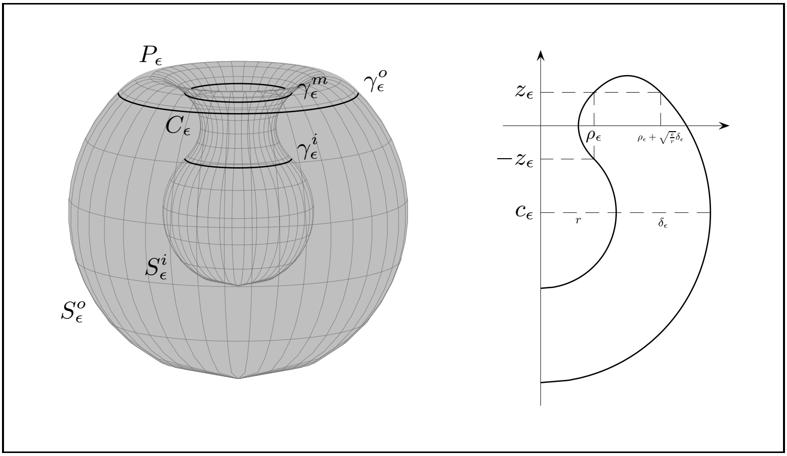

FIGURE 4.2.

Then

$$\tau_\epsilon(t) \in \left[-\frac{1}{\sqrt{\frac{r}{\epsilon}-1}}, \frac{1}{\sqrt{\frac{r}{\epsilon}-1}}\right] \qquad \text{for } t \in \left[-\rho_\epsilon - \sqrt{\frac{\epsilon}{r}}\delta_\epsilon, -\rho_\epsilon\right].$$

Consequently, in view of (4.8a),

$$\sqrt{\det g}\Big|_{P_\epsilon}(t) = -t\sqrt{1+\tau_\epsilon^2(t)} \leq \frac{r}{\sqrt{\frac{r}{\epsilon}-1}} + \frac{2z_\epsilon\sqrt{\frac{\epsilon}{r}}}{1-\frac{\epsilon}{r}} \qquad \text{for } t \in \left[-\rho_\epsilon - \sqrt{\frac{\epsilon}{r}}\delta_\epsilon, -\rho_\epsilon\right].$$

Thus

$$\begin{aligned} \operatorname{vol}_2(P_\epsilon) = \int_{P_\epsilon} 1 &\leq 2\pi\left(\frac{r}{\sqrt{\frac{r}{\epsilon}-1}} + \frac{2z_\epsilon\sqrt{\frac{\epsilon}{r}}}{1-\frac{\epsilon}{r}}\right)\sqrt{\frac{\epsilon}{r}}\frac{2z_\epsilon}{\sqrt{1-\frac{\epsilon}{r}}} \\ &= \frac{4\pi r z_\epsilon}{\frac{r}{\epsilon}-1} + \frac{8\pi\epsilon z_\epsilon^2}{r\left(1-\frac{\epsilon}{r}\right)^{3/2}} \xrightarrow{\epsilon\searrow 0} 0. \end{aligned}$$

Similarly,

$$\sqrt{\det g}\Big|_{C_\epsilon}(t) = \epsilon\cosh^2\left(\frac{t}{\epsilon}\right) \in [\epsilon, r] \qquad \text{for } t \in [-z_\epsilon, z_\epsilon],$$

and so

$$\operatorname{vol}_2(C_\epsilon) = \int_{C_\epsilon} 1 \leq 4\pi r\epsilon \operatorname{arccosh}\sqrt{\frac{r}{\epsilon}} \leq 4\pi r\epsilon\ln\left(2\sqrt{\frac{r}{\epsilon}}\right) = 2\pi r\epsilon\ln\left(\frac{4r}{\epsilon}\right) \xrightarrow{\epsilon\searrow 0} 0,$$

where we have used, once again,

$$\ln t \leq \operatorname{arccosh} t = \ln\left(t+\sqrt{t^2-1}\right) \leq \ln(2t) \qquad \forall t \geq 1.$$

With (4.14), we conclude (4.13b) and (4.13a). It remains to show (4.13c).

We begin with showing that $\lim_{\epsilon\searrow 0}\int_{\Sigma_\epsilon}|\mathring{A}|^2=8\pi$. Since $\mathring{A}\big|_{S_\epsilon^{i/o}}=0$, we infer from (4.14) that $\lim_{\epsilon\searrow 0}\int_{S_\epsilon^{i/o}}|\mathring{A}|^2=0$. On C_ϵ, (4.8b) and (4.8c) imply that

$$|\mathring{A}|^2\Big|_{C_\epsilon}(t)=\frac{2}{\epsilon^2\cosh^4\left(\frac{t}{\epsilon}\right)},\qquad t\in[-z_\epsilon,z_\epsilon].$$

Consequently, since $\sqrt{\det g}\Big|_{C_\epsilon}(t)=\epsilon\cosh^2\left(\frac{t}{\epsilon}\right)$,

$$\begin{aligned}\int_{C_\epsilon}|\mathring{A}|^2&=4\pi\int_{-\epsilon\operatorname{arccosh}\sqrt{\frac{r}{\epsilon}}}^{\epsilon\operatorname{arccosh}\sqrt{\frac{r}{\epsilon}}}\frac{dt}{\epsilon\cosh^2\left(\frac{t}{\epsilon}\right)}=8\pi\int_1^{\sqrt{\frac{r}{\epsilon}}}\frac{ds}{s^2\sqrt{s^2-1}}\\&=8\pi\left[\sqrt{1-\frac{1}{s}}\right]_{s=1}^{\sqrt{\frac{\epsilon}{r}}}=8\pi\sqrt{1-\frac{\epsilon}{r}}\xrightarrow{\epsilon\searrow 0}8\pi.\end{aligned}$$

Similarly, in view of (4.8a), (4.8b) and (4.8c), we have on P_ϵ,

$$|\mathring{A}|^2\sqrt{\det g}\Big|_{P_\epsilon}(t)=-\frac{t}{2\sqrt{1+\tau_\epsilon^2(t)}}\left(\frac{\tau_\epsilon(t)}{t}+\frac{1}{z_\epsilon\left(1+\tau_\epsilon^2(t)\right)}\right)^2,$$
$$t\in\left[-\rho_\epsilon-\sqrt{\frac{\epsilon}{r}}\delta_\epsilon,-\rho_\epsilon\right],$$

where, again, $\tau_\epsilon(t)$ is given by

$$\tau_\epsilon(t)=-\frac{t+\rho_\epsilon}{z_\epsilon}-\frac{1}{\sqrt{\frac{r}{\epsilon}-1}}.$$

For $t\in\left[-\rho_\epsilon-\sqrt{\frac{\epsilon}{r}}\delta_\epsilon,-\rho_\epsilon\right]$,

$$\frac{-t}{2\sqrt{1+\tau_\epsilon^2(t)}}\in\left[\frac{\sqrt{\epsilon}\sqrt{r-\epsilon}}{2},\frac{\sqrt{r\epsilon}}{2}+\frac{z_\epsilon}{\sqrt{\frac{r}{\epsilon}-1}}\right].$$

Moreover,

$$\frac{\tau_\epsilon(t)}{t}\in\left[-\frac{1}{2z_\epsilon+\sqrt{r^2-r\epsilon}},\frac{1}{\sqrt{r^2-r\epsilon}}\right]\qquad\text{and}\qquad\frac{1}{z_\epsilon\left(1+\tau_\epsilon^2(t)\right)}\in\left[\frac{1-\frac{\epsilon}{r}}{z_\epsilon},\frac{1}{z_\epsilon}\right],$$

so that

$$\frac{\tau_\epsilon(t)}{t}+\frac{1}{z_\epsilon\left(1+\tau_\epsilon^2(t)\right)}\in\left[\frac{1-\frac{\epsilon}{r}}{z_\epsilon}-\frac{1}{2z_\epsilon+\sqrt{r^2-r\epsilon}},\frac{1}{z_\epsilon}+\frac{1}{\sqrt{r^2-r\epsilon}}\right].$$

For ϵ small enough, the lower bound is non–negative, and we can estimate

$$|\mathring{A}|^2\sqrt{\det g}\Big|_{P_\epsilon}\leq\frac{\sqrt{r\epsilon}}{2z_\epsilon^2}+\frac{2}{z_\epsilon\sqrt{\frac{r}{\epsilon}-1}}+\frac{5}{2\sqrt{r\epsilon}\left(\frac{r}{\epsilon}-1\right)}+\frac{z_\epsilon}{r\epsilon\left(\frac{r}{\epsilon}-1\right)^{3/2}}.$$

Consequently,

$$\int_{P_\epsilon}|\mathring{A}|^2\leq\frac{2\pi r\epsilon}{\sqrt{r^2-r\epsilon}z_\epsilon}+\frac{8\pi r\epsilon}{r^2-r\epsilon}+\frac{10\pi r\epsilon z_\epsilon}{\left(r^2-r\epsilon\right)^{3/2}}+\frac{4\pi r\epsilon z_\epsilon^2}{\left(r^2-r\epsilon\right)^2}\xrightarrow{\epsilon\searrow 0}0,$$

and we conclude that, indeed,

$$\lim_{\epsilon\searrow 0}\int_{\Sigma_\epsilon}|\mathring{A}|^2 = 8\pi.$$

In order to establish that

$$\lim_{\epsilon\searrow 0}\int_{\Sigma_\epsilon}\left|A-\frac{\overline{H}}{2}g\right|^2 = 24\pi,$$

where $\overline{H}=\frac{1}{\mathrm{vol}_2(\Sigma_\epsilon)}\int_{\Sigma_\epsilon}H$, we proceed in a completely analogous manner. First, on P_ϵ, we get

$$\begin{aligned}\int_{P_\epsilon}H^2 &= 2\pi\int_{-\sqrt{r\epsilon}-\frac{2z_\epsilon}{\sqrt{\frac{r}{\epsilon}-1}}}^{-\sqrt{r\epsilon}}\frac{-t}{\sqrt{1+\tau_\epsilon^2(t)}}\left(-\frac{\tau_\epsilon(t)}{t}+\frac{1}{z_\epsilon\left(1+\tau_\epsilon^2(t)\right)}\right)^2 dt\\ &\le \frac{4\pi\epsilon\sqrt{r}}{\sqrt{r-\epsilon}}\Bigg(\frac{1}{z_\epsilon}+\frac{2}{\sqrt{r^2-r\epsilon}+2z_\epsilon}+\frac{1}{\sqrt{r^2-r\epsilon}}+\frac{z_\epsilon}{\left(\sqrt{r^2-r\epsilon}+2z_\epsilon\right)^2}\\ &\qquad\qquad +\frac{2z_\epsilon}{\left(\sqrt{r^2-r\epsilon}+2z_\epsilon\right)\sqrt{r^2-r\epsilon}}+\frac{z_\epsilon^2}{\left(\sqrt{r^2-r\epsilon}+2z_\epsilon\right)^2\sqrt{r^2-r\epsilon}}\Bigg)\\ &\xrightarrow{\epsilon\searrow 0} 0.\end{aligned}$$

Since $\lim_{\epsilon\searrow 0}\mathrm{vol}_2(P_\epsilon)=0$, this also implies

$$\lim_{\epsilon\searrow 0}\left|\int_{P_\epsilon}H\right|\le\lim_{\epsilon\searrow 0}\left(\mathrm{vol}_2(P_\epsilon)\int_{P_\epsilon}H^2\right)^{1/2}=0.$$

Next, on $S_\epsilon^{i/o}$, we calculate

$$\begin{aligned}\int_{S_\epsilon^i}H &= -4\pi\left(z_\epsilon-(c_\epsilon-r)\right)=-4\pi\left(2z_\epsilon+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right)\xrightarrow{\epsilon\searrow 0}-8\pi r,\\ \int_{S_\epsilon^i}H^2 &= \frac{8\pi}{r}\left(z_\epsilon-(c_\epsilon-r)\right)=\frac{8\pi}{r}\left(2z_\epsilon+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right)\xrightarrow{\epsilon\searrow 0}16\pi,\\ \int_{S_\epsilon^o}H &= 4\pi\left(z_\epsilon-(c_\epsilon-r-\delta_\epsilon)\right)\\ &= 4\pi\left(2z_\epsilon\left(1+\frac{1}{\sqrt{1-\frac{\epsilon}{r}}}\right)+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right)\xrightarrow{\epsilon\searrow 0}8\pi r\end{aligned}$$

and

$$\begin{aligned}\int_{S_\epsilon^o}H^2 &= \frac{8\pi}{r+\delta_\epsilon}\left(z_\epsilon-(c_\epsilon-r-\delta_\epsilon)\right)\\ &= \frac{8\pi}{r+\frac{2z_\epsilon}{\sqrt{1-\frac{\epsilon}{r}}}}\left(2z_\epsilon\left(1+\frac{1}{\sqrt{1-\frac{\epsilon}{r}}}\right)+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right)\xrightarrow{\epsilon\searrow 0}16\pi,\end{aligned}$$

where we have used that, on a sphere of radius R, $H = 2/R$ and $\sqrt{\det g} = R$. Finally, in view of $H\big|_{C_\epsilon} = 0$, we conclude that

$$\lim_{\epsilon \searrow 0} \int_{\Sigma_\epsilon} H = 0 \qquad \text{and} \qquad \lim_{\epsilon \searrow 0} \int_{\Sigma_\epsilon} H^2 = 32\pi.$$

Thus

$$\begin{aligned}\lim_{\epsilon \searrow 0} \int_{\Sigma_\epsilon} \left| A - \frac{\overline{H}}{2} g \right|^2 &= \lim_{\epsilon \searrow 0} \left(\int_{\Sigma_\epsilon} |\mathring{A}|^2 + \frac{1}{2} \int_{\Sigma_\epsilon} H^2 - \frac{1}{2} \overline{H} \int_{\Sigma_\epsilon} H \right) \\ &= 24\pi,\end{aligned}$$

as claimed. This establishes (4.13c). As in Section 2, a standard smoothing argument on the generating curves of the family Σ_ϵ yields the full statement of Proposition 4.6. □

Remark 4.7. *One might be tempted to think that, by the same technique, we could attach even more spheres inside the ones just constructed in order to obtain a bigger quotient in* (4.13c)*. However, a quick inspection reveals that*

$$\lim_{\epsilon \searrow 0} \frac{\displaystyle\int_{\Sigma_\epsilon^N} \left| A - \frac{1}{2\mathrm{vol}_2(\Sigma_\epsilon^N)} \left(\int_{\Sigma_\epsilon^N} H \right) g \right|^2}{\displaystyle\int_{\Sigma_\epsilon^N} |\mathring{A}|^2} = \begin{cases} 2, & N \textit{ odd}, \\[2ex] 2 + \frac{1}{N-1}, & N \textit{ even}, \end{cases}$$

if we arrange that Σ_ϵ^N converges to $N \geq 2$ copies of a sphere S_0 of radius r.

APPENDIX A

A few small lemmas

This appendix contains three little results that were used in the thesis but did not really fit anywhere else. In particular, we feel that stating and proving them at the locations where they were used would have broken too much the current train of thoughts. Also, we think that they might be interesting on their own right.

Contents

1. A Morrey–type estimate

In this section we want to show the following variation of Morrey's embedding theorem, valid for $W^{1,p}$–functions ($p > n$) on the open ball $B_R(0)$ of radius $R > 0$ around the origin in $\mathbb{R}^n$ ($n \geq 1$).

Lemma A.1. *If $R > 0$, $n < p \leq \infty$, $u \in W^{1,p}(B_R(0))$ and $\alpha = 1 - n/p$, then there is a constant C, depending only on n and p, such that*

$$\sup_{\substack{x,y\in B_R(0)\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C \, \|Du\|_{L^p(B_R(0))} \, .$$

Proof. Since $W^{1,p} \cap C^\infty$ is dense in $W^{1,p}$ for all p and on any domain (cf., e.g., [**Eva98**, Thm.2, §5.3.2, p.251]), we will henceforth assume that $u \in W^{1,p}(B_R(0)) \cap C^\infty(B_R(0))$. The usual Morrey estimate is as follows (see, e.g., Corollary IX.14 on p.168 in [**Bre83**]):

$$\sup_{x\in B_R(0)} |u(x)| + \sup_{\substack{x,y\in B_R(0)\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq \overline{C}_R \left(\|u\|_{L^p(B_R(0))} + \|Du\|_{L^p(B_R(0))} \right) ,$$

where $\overline{C}_R$ is a constant independent of u, but depending on the radius R of the ball under consideration (as well as on n and p). The Lemma therefore states that the Hölder semi–norm of u can be bounded only by the L^p norm of its derivative, and that this can be done independently of the size of the ball u is defined on. The proof of this is done by a scaling argument.

Set $v(x) = u(Rx)$. Then $v \in W^{1,p}(B_1(0)) \cap C^\infty(B_1(0))$. Moreover, we have that

$$
\begin{aligned}
\sup_{x \in B_1(0)} |v(x)| &= \sup_{x \in B_R(0)} |u(x)|, \\
\sup_{\substack{x,y \in B_1(0) \\ x \neq y}} \frac{|v(x) - v(y)|}{|x-y|^\alpha} &= \sup_{\substack{x,y \in B_1(0) \\ x \neq y}} \frac{|u(Rx) - u(Ry)|}{|Rx - Ry|^\alpha} R^\alpha \\
&= R^\alpha \sup_{\substack{x,y \in B_R(0) \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \\
\|v\|_{L^p(B_1(0))} &= \left(\int_{B_1(0)} |v(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= \left(\frac{1}{R^n} \int_{B_1(0)} |u(Rx)|^p \, d(Rx) \right)^{\frac{1}{p}} \\
&= R^{-\frac{n}{p}} \|u\|_{L^p(B_R(0))}, \\
\|Dv\|_{L^p(B_1(0))} &= \left(\int_{B_1(0)} |Dv(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= \left(\frac{1}{R^n} \int_{B_1(0)} \Big| \underbrace{D\big(u(Rx)\big)}_{= R(Du)(Rx)} \Big|^p \, d(Rx) \right)^{\frac{1}{p}} \\
&= \left(R^{p-n} \int_{B_R(0)} |Du(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= R^{1-\frac{n}{p}} \|Du\|_{L^p(B_R(0))}.
\end{aligned}
$$

Now, denoting by $\widetilde{\omega}_n = \mathrm{vol}_n\big(B_1(0)\big)$, we let

$$
\begin{aligned}
\widetilde{u}(x) &= u(x) - \frac{1}{\widetilde{\omega}_n R^n} \int_{B_R(0)} u(x') \, dx', \\
\widetilde{v}(x) &= v(x) - \frac{1}{\widetilde{\omega}_n} \int_{B_1(0)} v(x') \, dx'.
\end{aligned}
$$

Notice that $\widetilde{u} \in W^{1,p}(B_R(0)) \cap C^\infty(B_R(0))$ and $\widetilde{v} \in W^{1,p}(B_1(0)) \cap C^\infty(B_1(0))$. Thus, the usual Morrey embedding yields:

$$
\sup_{\substack{x,y \in B_1(0) \\ x \neq y}} \frac{|\widetilde{v}(x) - \widetilde{v}(y)|}{|x-y|^\alpha} \leq \overline{C}_1 \left(\|\widetilde{v}\|_{L^p(B_1(0))} + \|D\widetilde{v}\|_{L^p(B_1(0))} \right),
$$

where the constant $\overline{C}_1$ is independent of $\widetilde{v}$ and of R. By construction, $\widetilde{v}(x)-\widetilde{v}(y)=v(x)-v(y)$ and $D\widetilde{v}=Dv$. Moreover, by our considerations above, $\|\widetilde{v}\|_{L^p(B_1(0))}=R^{-n/p}\|\widetilde{u}\|_{L^p(B_R(0))}$, since $\widetilde{v}(x)=\widetilde{u}(Rx)$. Applying the Poincaré Lemma for balls (cf., e.g., [**Eva98**, Thm.2, §5.8.1, p.276]), we obtain $\|\widetilde{u}\|_{L^p(B_R(0))}\le C_P R\|Du\|_{L^p(B_R(0))}$, where C_P is independent of u and R. Putting all this together, we finally arrive at:

$$R^\alpha \sup_{\substack{x,y\in B_R(0)\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le \overline{C}_1(1+C_P)R^{1-\frac{n}{p}}\|Du\|_{L^p(B_R(0))},$$

which is the desired inequality, since $\alpha=1-n/p$. □

2. On the restriction of the second fundamental form to a linear subspace

In this section we wish to prove that the second fundamental form of the intersection of a convex hypersurface with a linear subspace is controlled in each point by the second fundamental form of the hypersurface and the angle between the Gauss map of the hypersurface and the linear subspace.

Let $n\ge 2$ and $\Omega\subset\mathbb{R}^{n+1}$ be an open convex set with smooth boundary $\Sigma=\partial\Omega$. Assume we are given a k–dimensional ($k\in\{2,\dots,n\}$) linear subspace Π_x^k of $\mathbb{R}^{n+1}$ passing through $x\in\Omega$. Set $\overline{\Sigma}=\Sigma\cap\Pi_x^k$, and let ν and $\overline{\nu}$ denote the Gauss-maps of Σ in $\mathbb{R}^{n+1}$ and $\overline{\Sigma}$ in Π_x^k, respectively. Also, let A and $\overline{A}$ be the second fundamental forms of Σ in $\mathbb{R}^{n+1}$ and $\overline{\Sigma}$ in Π_x^k, respectively. Finally, for each $q\in\overline{\Sigma}\subset\Sigma$, let $\alpha(q)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ denote the angle between ν in q and Π_x^k. We have $\cos\alpha(q)=\langle\nu(q),\overline{\nu}(q)\rangle_{\mathbb{R}^{n+1}}$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n+1}}$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. Moreover, $\cos\alpha(q)\neq 0$ (i.e., $\alpha(q)\notin\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$), since x was an inner point of Ω. The following holds.

Lemma A.2. *In every point $q\in\overline{\Sigma}$, we have*

$$\left|\overline{A}_q\right|\le\frac{1}{\cos\alpha(q)}\left|A_q\right|.$$

PROOF. Without loss of generality, we may assume that $x=0$. Since all the considerations that follow are valid pointwise, we will drop subscripts or other references to the point at hand. Notice that, in each point $q\in\overline{\Sigma}=\Sigma\cap\Pi_0^k$, we can write

$$\nu=\cos\alpha\overline{\nu}+\nu^\perp,$$

where $\nu^\perp\in\left(\Pi_0^k\right)^\perp$ is a vector in the orthogonal complement of Π_0^k. Let $\overline{X}$ and $\overline{Y}$ be two arbitrary vector fields tangent to $\overline{\Sigma}$ and extended, first to vector fields tangent to Σ, and then to vector fields in $\mathbb{R}^{n+1}$, each time in a relative neighbourhood of $\overline{\Sigma}$. Denoting by D the standard derivation in $\mathbb{R}^{n+1}$, we have, in every point of $\overline{\Sigma}$:

$$\begin{aligned} A(\overline{X},\overline{Y}) &= -\left\langle D_{\overline{X}}\overline{Y},\nu\right\rangle_{\mathbb{R}^{n+1}} = -\cos\alpha\left\langle D_{\overline{X}}\overline{Y},\overline{\nu}\right\rangle_{\mathbb{R}^{n+1}} - \left\langle D_{\overline{X}}\overline{Y},\nu^\perp\right\rangle_{\mathbb{R}^{n+1}} \\ &= -\cos\alpha\left\langle D_{\overline{X}}\overline{Y},\overline{\nu}\right\rangle_{\mathbb{R}^{n+1}} = \cos\alpha\overline{A}(\overline{X},\overline{Y}), \end{aligned}$$

where we have used that $D_{\overline{X}}\overline{Y} \in \Pi_0^k$ if $\overline{X}, \overline{Y} \in \Pi_0^k$.
Now fix $q \in \overline{\Sigma}$. If we choose an orthonormal basis $(\epsilon_1, \ldots, \epsilon_n)$ in $T_q\Sigma$ such that $(\epsilon_1, \ldots, \epsilon_k)$ is an orthonormal basis in $T_q\overline{\Sigma}$ that diagonalises $\overline{A}$, we have

$$\begin{aligned}\left|\overline{A}\right|^2 &= \sum_{i=1}^{k} \left(\overline{A}(\epsilon_i, \epsilon_i)\right)^2 = \frac{1}{\cos^2\alpha} \sum_{i=1}^{k} \left(A(\epsilon_i, \epsilon_i)\right)^2 \\ &\leq \frac{1}{\cos^2\alpha} \sum_{j=1}^{n}\sum_{i=1}^{k} \left(A(\epsilon_i, \epsilon_j)\right)^2 \leq \frac{1}{\cos^2\alpha} \sum_{i,j=1}^{n} \left(A(\epsilon_i, \epsilon_j)\right)^2 = \frac{1}{\cos^2\alpha} \left|A\right|^2 .\end{aligned}$$

The lemma follows. □

3. On the variation of the Gauss map along a curve in a convex hypersurface

In this section we establish that the difference of the Gauss map of a convex hypersurface Σ between two points can be estimated in magnitude from above by the integral of the largest principal curvature of Σ along any curve in Σ joining those two points.

The following considerations are valid for any dimension $n \geq 1$. $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$, and Greek indices refer to components in the usual basis of $\mathbb{R}^{n+1}$. For an immersion $f : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, Latin indices refer to components in the induced coordinate basis and D denotes differentiation with respect to this basis.

Assume $\Sigma = \partial\Omega \subset \mathbb{R}^{n+1}$ is a smooth hypersurface, where Ω is an open convex domain. Since Ω is convex, we can parametrise Σ by S^n (through projection). Moreover (using stereographic projection), we can pick a point $S \in \Sigma$ such that there is a set $U \subset \mathbb{R}^n$ and an embedding $f : U \to \mathbb{R}^{n+1}$ with $f(U) = \Sigma \setminus \{S\}$. Let g denote the Riemannian metric of Σ. Let A denote the second fundamental form of Σ with eigenvalues $\lambda_n \geq \ldots \geq \lambda_1 \geq 0$ and corresponding orthonormal eigenframe $(\epsilon_1, \ldots, \epsilon_n)$ (i.e. $\sum_{j=1}^n A^i{}_j (\epsilon_l)^j = \lambda_l (\epsilon_l)^i$). Finally, let $\nu : U \to \mathbb{R}^{n+1}$ be the Gauss map which associates to each coordinate value x the outer unit normal vector to Σ based at $f(x)$. Then the following is true:

Lemma A.3. *For any points $q_1, q_2 \in U$ and any path $\gamma : [0,1] \to U$ joining them (i.e. with $\gamma(0) = q_1$ and $\gamma(1) = q_2$), we have that*

$$|\nu(q_2) - \nu(q_1)| \leq \int_\gamma \lambda_n.$$

Taking into consideration the pointwise inequality

$$\lambda_n = \sqrt{\lambda_n^2} \leq \sqrt{\sum_{j=1}^{n} \lambda_j^2} = |A| ,$$

we also have

Corollary A.4. *If U, q_1, q_2 and γ are as in the Lemma, then*

$$|\nu(q_2) - \nu(q_1)| \leq \int_\gamma |A|\,.$$

PROOF OF LEMMA A.3. For q_1, q_2 and γ as above, we have

$$\big(\nu(q_2)\big)^\alpha - \big(\nu(q_1)\big)^\alpha = \int_0^1 \sum_{i=1}^n \Big(\big(D_i\nu\big)^\alpha \circ \gamma(t)\Big)\dot\gamma^i(t)\,dt.$$

Thus

$$|\nu(q_2) - \nu(q_1)| \leq \int_0^1 \left| \sum_{i=1}^n \Big(\big(D_i\nu\big) \circ \gamma(t)\Big)\dot\gamma^i(t)\right| dt.$$

Consider for any $i, l \in \{1, \dots, n\}$ the pointwise equality

$$\begin{aligned}\left\langle D_i f, \sum_{j=1}^n (\epsilon_l)^j D_j\nu \right\rangle &= -\sum_{j=1}^n A_{ij}(\epsilon_l)^j = -\sum_{j,k=1}^n g_{ik}A^k{}_j(\epsilon_l)^j = -\sum_{k=1}^n g_{ik}(\lambda_l)(\epsilon_l)^k \\ &= \left\langle D_i f, -\sum_{k=1}^n (\lambda_l)(\epsilon_l)^k D_k f\right\rangle.\end{aligned}$$

Since the normal part of $D_j\nu$ vanishes ($0 = D_j 1 = D_j\langle\nu,\nu\rangle = 2\langle\nu, D_j\nu\rangle$), we obtain

$$\sum_{j=1}^n (\epsilon_l)^j D_j\nu = \sum_{k=1}^n (-\lambda_l)(\epsilon_l)^k D_k f \qquad (\forall l).$$

Expanding $\dot\gamma$ in the orthonormal eigenbasis $(\epsilon_l)_{l\in\{1,\dots,n\}}$ of A,

$$\dot\gamma^i = \sum_{l,r,s=1}^n g_{rs}\dot\gamma^r(\epsilon_l)^s(\epsilon_l)^i,$$

we obtain

$$\begin{aligned}\left|\sum_{i=1}^n \dot\gamma^i D_i\nu\right|^2 &= \sum_{i,j=1}^n \left\langle \dot\gamma^i D_i\nu, \dot\gamma^j D_j\nu\right\rangle = \sum_{i,j,l,r,s=1}^n \dot\gamma^j \left\langle g_{rs}\dot\gamma^r(\epsilon_l)^s(\epsilon_l)^i D_i\nu, D_j\nu\right\rangle \\ &= \sum_{j,k,l,r,s=1}^n \dot\gamma^j \left\langle (-\lambda_l) g_{rs}\dot\gamma^r(\epsilon_l)^s(\epsilon_l)^k D_k f, D_j\nu\right\rangle \\ &= \sum_{j,k,l,r,s=1}^n \dot\gamma^j(\lambda_l) g_{rs}\dot\gamma^r(\epsilon_l)^s(\epsilon_l)^k \left(-\left\langle D_k f, D_j\nu\right\rangle\right) \\ &= \sum_{j,k,l,r,s=1}^n \dot\gamma^j(\lambda_l) g_{rs}\dot\gamma^r(\epsilon_l)^s(\epsilon_l)^k A_{jk} \\ &= \sum_{j,k,l,r,s=1}^n \dot\gamma^j(\lambda_l)^2 g_{rs}\dot\gamma^r(\epsilon_l)^s(\epsilon_l)^k g_{jk}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k,l,r,s=1}^{n} (\lambda_l)^2 g_{rs}\dot{\gamma}^r (\epsilon_l)^s g_{jk}\dot{\gamma}^j (\epsilon_l)^k = \sum_{l=1}^{n} (\lambda_l)^2 \Big(g\left(\dot{\gamma}, \epsilon_l\right) \Big)^2 \\
&\leq \sum_{l=1}^{n} (\lambda_n)^2 \Big(g\left(\dot{\gamma}, \epsilon_l\right) \Big)^2 \\
&= (\lambda_n)^2 \sum_{j,k,l,r,s=1}^{n} g_{rs}\dot{\gamma}^r (\epsilon_l)^s g_{jk}\dot{\gamma}^j (\epsilon_l)^k \\
&= (\lambda_n)^2 \sum_{j,k,r,s=1}^{n} \sum_{l=1}^{n} \sum_{l'=1}^{n} \delta_{ll'} g_{rs}\dot{\gamma}^r (\epsilon_l)^s g_{jk}\dot{\gamma}^j (\epsilon_{l'})^k \\
&= (\lambda_n)^2 \sum_{j,k,r,s=1}^{n} \sum_{l=1}^{n} \sum_{l'=1}^{n} g(\epsilon_l, \epsilon_{l'}) g_{rs}\dot{\gamma}^r (\epsilon_l)^s g_{jk}\dot{\gamma}^j (\epsilon_{l'})^k \\
&= (\lambda_n)^2 g \left(\sum_{l,r,s=1}^{n} g_{rs}\dot{\gamma}^r (\epsilon_l)^s \epsilon_l \,,\, \sum_{j,k,l'=1}^{n} g_{jk}\dot{\gamma}^j (\epsilon_{l'})^k \epsilon_{l'} \right) \\
&= (\lambda_n)^2 g(\dot{\gamma}, \dot{\gamma}),
\end{aligned}
$$

where the inequality follows from $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. Consequently,

$$
\left| \sum_{i=1}^{n} \dot{\gamma}^i D_i \nu \right| \leq \lambda_n \left\| \dot{\gamma} \right\|_g = \lambda_n \sqrt{g(\dot{\gamma}, \dot{\gamma})}.
$$

This yields

$$
|\nu(q_2) - \nu(q_1)| \leq \int_0^1 \Big(\lambda_n \circ \gamma(t) \Big) \left\| \dot{\gamma}(t) \right\|_{g \circ \gamma(t)} \, dt = \int_{\gamma} \lambda_n .
$$

$\square$

APPENDIX B

On the second variation around a spherical cap of some L^2–integral quantities along a volume–preserving flow

This appendix contains some computations made in a first attempt to combine the results of the present thesis with those of [**DLT10**]. More precisely, we consider a spherical cap M with (totally–umbilical) boundary ∂M, and deform its metric in a volume preserving manner (thus deforming the induced metric on ∂M, as well). On M, we then consider the first and second variations of the L^2–norms of the traceless Ricci tensor, as well as of the difference between the scalar curvature and its mean across M. Afterwards, we do the same on ∂M with the L^2–norms of the traceless Ricci tensor, as well as of the induced quantities $\mathring{A}$ and $H - \overline{H}$ ($\overline{H}$ denoting the mean of H across ∂M). We end by exposing a simpler situation.

The four involved quantities are precisely those relevant in [**DLT10**] and in this work.

Contents

Consider a closed subset M of the $(n+1)$–sphere S^{n+1}_R with radius R, such that its boundary ∂M coincides with an n–sphere S^n_r of radius r. We want to look at metric deformations of S^{n+1}_R which fix the volume of M. So, denoting by $\widehat{\eta}$ the canonical metric of S^{n+1}_R, we consider the one–parameter family $\widehat{g}_t$ of metrics on S^{n+1}_R which satisfies (at least locally around $t = 0$)

$$\begin{cases} \partial_t \widehat{g}_t &= \widehat{h}, \\ \widehat{g}_0 &= \widehat{\eta}, \end{cases}$$

where $\widehat{h}$ denotes a symmetric bilinear form on S^{n+1}_R such that

$$\int_M \operatorname{tr}_{\widehat{g}_t} \widehat{h} \, d\mathrm{vol}_{\widehat{g}_t} = 0. \tag{B.1}$$

Let g_t, η and h denote the restrictions of $\widehat{g}_t$, $\widehat{\eta}$ and $\widehat{h}$ to M, respectively. Let $\widetilde{g}_t$ and $\widetilde{\eta}$ denote the metrics induced by g_t and η on ∂M, respectively. In what follows, we wish to calculate the second variation with respect to the preceding flow of the following L^2–integral quantities on M and ∂M which are derived from the respective metrics g_t and $\widetilde{g}_t$ and which are critical at $t = 0$.

$$\int_M \|\overset{\circ}{\mathrm{Ric}}\|_g^2 \qquad \text{(see eqs. (B.19) and (B.42))},$$

$$\int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)^2 \qquad \text{(see eqs. (B.21) and (B.43))},$$

$$\int_{\partial M} \|\overset{\circ}{A}\|_{\widetilde{g}}^2 \qquad \text{(see eqs. (B.38)) and (B.44))},$$

$$\int_{\partial M} \left(H - \overline{H}\right)^2 \qquad \text{(see eqs. (B.40)) and (B.45))},$$

$$\int_{\partial M} \|\overset{\circ}{\mathrm{Ric}}\|_{\widetilde{g}}^2 \qquad \text{(see eqs. (B.41) and (B.46))}.$$

We start by setting up our notations and conventions.

1. Notations and conventions

For simplicity, we will suppress any explicit reference to the dependence of a quantity on t. The following notations are used for some quantities derived from the metric g:

Γ	Christoffel symbols of the second kind
Riem	Riemann tensor
Ric	Ricci tensor
$\overset{\circ}{\mathrm{Ric}}$	Traceless part of Ric
Scal	Scalar curvature
Vol	Volume of M with respect to g
∇	Levi–Civita connection
Δ	Laplace–Beltrami operator with respect to ∇
div	Covariant divergence operator acting on symmetric two–tensor fields or vector fields
∂	Coordinate derivative or coordinate vector fields

Here, we choose the following sign convention for the Riemannian curvature tensor:

$$\mathrm{Riem}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X - \nabla_X Y} Z,$$

where X, Y and Z are vector fields on M. This way, the $(n+1)$–sphere of radius R has positive scalar curvature

$$\mathrm{Scal}_{\widehat{\eta}} = \frac{n(n+1)}{R^2}.$$

The boundary ∂M of M will be viewed as an embedded submanifold of M, its induced metric will be denoted by $\widetilde{g}$, and the quantities from above will be equipped with a tilde to designate those respective to $(\partial M, \widetilde{g})$. Furthermore, whenever we work in local coordinates $x^0, \ldots, x^n$ near the boundary, we shall assume that x_0 vanishes on ∂M. Greek indices will run from 0 to n, whereas Latin ones will run from 1 to n. Indices will always be raised and lowered using the ambient metric g and summation over repeated ones shall be understood implicitly. We introduce the following notation for the extrinsic quantities inferred from the embedding of ∂M into M:

ν	Outer unit normal on ∂M
A	Second fundamental form
$\mathring{A}$	Traceless part of A
H	Mean curvature

Here, we choose the following sign convention for the second fundamental form:

$$A(\widetilde{X}, \widetilde{Y}) = -g\left(\nabla_{\widetilde{X}} \widetilde{Y}, \nu\right),$$

where $\widetilde{X}$ and $\widetilde{Y}$ are vector fields on ∂M (extended to a neighbourhood of ∂M in M). (Notice that in the case of a subset Ω of $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$, this sign convention for the second fundamental form of the boundary ensures the equivalence of its positive definiteness with the convexity of the enclosed volume Ω.)

Next, we introduce some notation for two–tensor fields B and C on M:

$B{:}C$	Full contraction of B and C with respect to g, i.e. $B{:}C = B^{\mu\nu} C_{\mu\nu}$
$\operatorname{tr}_g B$	Trace of B with respect to g, i.e. $\operatorname{tr}_g B = g{:}B$
$\lVert B \rVert_g$	Norm of B with respect to g, i.e. $\lVert B \rVert_g^2 = B{:}B$

Similarly, we define for two–tensor fields $\widetilde{B}$ and $\widetilde{C}$ on ∂M:

$\widetilde{B}\widetilde{:}\widetilde{C}$	Full contraction of $\widetilde{B}$ and $\widetilde{C}$ with respect to $\widetilde{g}$, i.e. $\widetilde{B}\widetilde{:}\widetilde{C} = \widetilde{B}_{ij} \left(\widetilde{g}^{-1}\right)^{ik} \left(\widetilde{g}^{-1}\right)^{jl} \widetilde{C}_{kl}$
$\operatorname{tr}_{\widetilde{g}} \widetilde{B}$	Trace of $\widetilde{B}$ with respect to $\widetilde{g}$, i.e. $\operatorname{tr}_{\widetilde{g}} \widetilde{B} = \widetilde{g}\widetilde{:}\widetilde{B}$
$\lVert \widetilde{B} \rVert_{\widetilde{g}}$	Norm of $\widetilde{B}$ with respect to $\widetilde{g}$, i.e. $\lVert \widetilde{B} \rVert_{\widetilde{g}}^2 = \widetilde{B}\widetilde{:}\widetilde{B}$

Finally, the following integral notations will be used for the average of (smooth) functions f over (M, g) and φ over $(\partial M, \widetilde{g})$, respectively:

$$\overline{f} = \fint_M f \, d\mathrm{vol}_g = \frac{1}{\mathrm{Vol}} \int_M f \, d\mathrm{vol}_g,$$
$$\overline{\varphi} = \fint_{\partial M} \varphi \, d\mathrm{vol}_{\widetilde{g}} = \frac{1}{\widetilde{\mathrm{Vol}}} \int_{\partial M} \varphi \, d\mathrm{vol}_{\widetilde{g}}.$$

2. The first and second variations of the quantities on M

We start from the evolution of g:

$$\partial_t g_{\mu\nu} = h_{\mu\nu}. \tag{B.2}$$

Then, denoting by δ the Kronecker delta,

$$
\begin{aligned}
& 0 = \partial_t \delta^{\mu}{}_{\lambda} = \partial_t \left(g^{\mu\nu} g_{\nu\lambda} \right) \overset{\text{(B.2)}}{=} \left(\partial_t g^{\mu\nu} \right) g_{\nu\lambda} + g^{\mu\nu} h_{\nu\lambda} \\
\Longrightarrow \quad & \partial_t g^{\mu\nu} = -h^{\mu\nu}. \qquad \text{(B.3)}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\partial_t \sqrt{\det g} &= \frac{\partial_t \det g}{2\sqrt{\det g}} = \frac{\det g}{2\sqrt{\det g}} \operatorname{tr}_g \left(\partial_t g \right) \\
&\overset{\text{(B.2)}}{=} \frac{1}{2} \operatorname{tr}_g h \sqrt{\det g}. \qquad \text{(B.4)}
\end{aligned}
$$

Therefore,

$$
\partial_t \mathrm{Vol} = \partial_t \int_M d\mathrm{vol}_g \overset{\text{(B.4)}}{=} \frac{1}{2} \int_M \operatorname{tr}_g h \, d\mathrm{vol}_g \overset{\text{(B.1)}}{=} 0. \qquad \text{(B.5)}
$$

Furthermore,

$$
\begin{aligned}
\partial_t \Gamma^{\sigma}_{\alpha\beta} &= \partial_t \left(\frac{1}{2} g^{\sigma\tau} \left(\partial_\alpha g_{\tau\beta} + \partial_\beta g_{\tau\alpha} - \partial_\tau g_{\alpha\beta} \right) \right) \\
&\overset{\text{(B.2),(B.3)}}{=} -\frac{1}{2} h^{\sigma\tau} \left(\partial_\alpha g_{\tau\beta} + \partial_\beta g_{\tau\alpha} - \partial_\tau g_{\alpha\beta} \right) + \frac{1}{2} g^{\sigma\tau} \left(\partial_\alpha h_{\tau\beta} + \partial_\beta h_{\tau\alpha} - \partial_\tau h_{\alpha\beta} \right) \\
&= \frac{1}{2} g^{\sigma\tau} \left(\nabla_\alpha h_{\tau\beta} + \nabla_\beta h_{\tau\alpha} - \nabla_\tau h_{\alpha\beta} \right) \\
&\quad + \frac{1}{2} g^{\sigma\tau} \left(\Gamma^{\rho}_{\alpha\tau} h_{\rho\beta} + \Gamma^{\rho}_{\alpha\beta} h_{\tau\rho} + \Gamma^{\rho}_{\beta\tau} h_{\rho\alpha} + \Gamma^{\rho}_{\beta\alpha} h_{\tau\rho} - \Gamma^{\rho}_{\tau\alpha} h_{\rho\beta} - \Gamma^{\rho}_{\tau\beta} h_{\alpha\rho} \right) \\
&\quad - \frac{1}{2} h^{\sigma}{}_{\rho} g^{\rho\tau} \left(\partial_\alpha g_{\tau\beta} + \partial_\beta g_{\tau\alpha} - \partial_\tau g_{\alpha\beta} \right) \\
&= \frac{1}{2} g^{\sigma\tau} \left(\nabla_\alpha h_{\tau\beta} + \nabla_\beta h_{\tau\alpha} - \nabla_\tau h_{\alpha\beta} \right) + h^{\sigma}{}_{\rho} \Gamma^{\rho}_{\alpha\beta} - h^{\sigma}{}_{\rho} \Gamma^{\rho}_{\alpha\beta} \\
&= \frac{1}{2} g^{\sigma\tau} \left(\nabla_\alpha h_{\tau\beta} + \nabla_\beta h_{\tau\alpha} - \nabla_\tau h_{\alpha\beta} \right). \qquad \text{(B.6)}
\end{aligned}
$$

From this,

$$
\begin{aligned}
\partial_t \operatorname{Riem}_{\alpha\beta\gamma}{}^{\sigma} &= \partial_t \left(\partial_\beta \Gamma^\sigma_{\alpha\gamma} - \partial_\alpha \Gamma^\sigma_{\beta\gamma} + \Gamma^\sigma_{\beta\mu}\Gamma^\mu_{\alpha\gamma} - \Gamma^\sigma_{\alpha\mu}\Gamma^\mu_{\beta\gamma}\right) \\
&= \partial_\beta \partial_t \Gamma^\sigma_{\alpha\gamma} - \partial_\alpha \partial_t \Gamma^\sigma_{\beta\gamma} \\
&\quad + \left(\partial_t \Gamma^\sigma_{\beta\mu}\right)\Gamma^\mu_{\alpha\gamma} + \Gamma^\sigma_{\beta\mu}\left(\partial_t \Gamma^\mu_{\alpha\gamma}\right) - \left(\partial_t \Gamma^\sigma_{\alpha\mu}\right)\Gamma^\mu_{\beta\gamma} - \Gamma^\sigma_{\alpha\mu}\left(\partial_t \Gamma^\mu_{\beta\gamma}\right) \\
&= \nabla_\beta \partial_t \Gamma^\sigma_{\alpha\gamma} - \nabla_\alpha \partial_t \Gamma^\sigma_{\beta\gamma} \\
&\quad + \left(\partial_t \Gamma^\sigma_{\mu\gamma}\right)\Gamma^\mu_{\beta\alpha} + \left(\partial_t \Gamma^\sigma_{\alpha\mu}\right)\Gamma^\mu_{\beta\gamma} - \left(\partial_t \Gamma^\mu_{\alpha\gamma}\right)\Gamma^\sigma_{\beta\mu} \\
&\quad - \left(\partial_t \Gamma^\sigma_{\mu\gamma}\right)\Gamma^\mu_{\alpha\beta} - \left(\partial_t \Gamma^\sigma_{\beta\mu}\right)\Gamma^\mu_{\alpha\gamma} + \left(\partial_t \Gamma^\mu_{\beta\gamma}\right)\Gamma^\sigma_{\alpha\mu} \\
&\quad + \left(\partial_t \Gamma^\sigma_{\beta\mu}\right)\Gamma^\mu_{\alpha\gamma} + \Gamma^\sigma_{\beta\mu}\left(\partial_t \Gamma^\mu_{\alpha\gamma}\right) - \left(\partial_t \Gamma^\sigma_{\alpha\mu}\right)\Gamma^\mu_{\beta\gamma} - \Gamma^\sigma_{\alpha\mu}\left(\partial_t \Gamma^\mu_{\beta\gamma}\right) \\
&= \nabla_\beta \partial_t \Gamma^\sigma_{\alpha\gamma} - \nabla_\alpha \partial_t \Gamma^\sigma_{\beta\gamma} \\
&\overset{(B.6)}{=} \frac{1}{2}\nabla_\beta \left(g^{\sigma\tau}\left(\nabla_\alpha h_{\tau\gamma} + \nabla_\gamma h_{\tau\alpha} - \nabla_\tau h_{\alpha\gamma}\right)\right) \\
&\quad - \frac{1}{2}\nabla_\alpha \left(g^{\sigma\tau}\left(\nabla_\beta h_{\tau\gamma} + \nabla_\gamma h_{\tau\beta} - \nabla_\tau h_{\beta\gamma}\right)\right) \\
&= \frac{1}{2} g^{\sigma\tau}\Big(\nabla_\beta \nabla_\alpha h_{\tau\gamma} - \nabla_\alpha \nabla_\beta h_{\tau\gamma} + \nabla_\beta \nabla_\gamma h_{\alpha\tau} \\
&\qquad - \nabla_\alpha \nabla_\gamma h_{\beta\tau} - \nabla_\beta \nabla_\tau h_{\alpha\gamma} + \nabla_\alpha \nabla_\tau h_{\beta\gamma}\Big).
\end{aligned}
$$

Applying the definition of the Riemann tensor along with the first Bianchi identities, we obtain:

$$
\begin{aligned}
\partial_t \operatorname{Riem}_{\alpha\beta\gamma}{}^{\sigma} &= \frac{1}{2} g^{\sigma\tau}\Big(-\operatorname{Riem}_{\alpha\beta\tau}{}^{\mu} h_{\mu\gamma} - \operatorname{Riem}_{\alpha\beta\gamma}{}^{\mu} h_{\mu\tau} \\
&\qquad + \nabla_\gamma \nabla_\beta h_{\alpha\tau} - \operatorname{Riem}_{\gamma\beta\alpha}{}^{\mu} h_{\mu\tau} - \operatorname{Riem}_{\gamma\beta\tau}{}^{\mu} h_{\alpha\mu} \\
&\qquad - \nabla_\alpha \nabla_\gamma h_{\beta\tau} - \nabla_\beta \nabla_\tau h_{\alpha\gamma} + \nabla_\alpha \nabla_\tau h_{\beta\gamma}\Big) \\
&= \frac{1}{2} g^{\sigma\tau}\Big(\operatorname{Riem}_{\beta\tau\alpha}{}^{\mu} h_{\mu\gamma} + \operatorname{Riem}_{\tau\alpha\beta}{}^{\mu} h_{\mu\gamma} - \operatorname{Riem}_{\alpha\beta\gamma}{}^{\mu} h_{\mu\tau} \\
&\qquad + \operatorname{Riem}_{\beta\alpha\gamma}{}^{\mu} h_{\mu\tau} + \operatorname{Riem}_{\alpha\gamma\beta}{}^{\mu} h_{\mu\tau} + \operatorname{Riem}_{\beta\gamma\tau}{}^{\mu} h_{\mu\alpha} \\
&\qquad + \nabla_\gamma \nabla_\beta h_{\tau\alpha} - \nabla_\alpha \nabla_\gamma h_{\beta\tau} - \nabla_\beta \nabla_\tau h_{\alpha\gamma} + \nabla_\alpha \nabla_\tau h_{\beta\gamma}\Big) \\
&= \frac{1}{2} g^{\sigma\tau}\Big(-2\operatorname{Riem}_{\alpha\beta\gamma}{}^{\mu} h_{\mu\tau} + \operatorname{Riem}_{\alpha\gamma\beta}{}^{\mu} h_{\mu\tau} + \operatorname{Riem}_{\beta\tau\alpha}{}^{\mu} h_{\mu\gamma} \\
&\qquad - \operatorname{Riem}_{\alpha\tau\beta}{}^{\mu} h_{\mu\gamma} - \operatorname{Riem}_{\gamma\beta\tau}{}^{\mu} h_{\mu\alpha} \\
&\qquad - \nabla_\alpha \nabla_\gamma h_{\beta\tau} - \nabla_\beta \nabla_\tau h_{\alpha\gamma} + \nabla_\alpha \nabla_\tau h_{\beta\gamma} + \nabla_\gamma \nabla_\beta h_{\tau\alpha}\Big).
\end{aligned}
\tag{B.7}
$$

Consequently,

$$\begin{aligned}
\partial_t \operatorname{Riem}_{\alpha\beta\gamma\delta} &= (\partial_t g_{\delta\sigma}) \operatorname{Riem}_{\alpha\beta\gamma}{}^{\sigma} + g_{\delta\sigma} \left(\partial_t \operatorname{Riem}_{\alpha\beta\gamma}{}^{\sigma}\right) \\
&\overset{(B.2),(B.7)}{=} h_{\delta\sigma} \operatorname{Riem}_{\alpha\beta\gamma}{}^{\sigma} \\
&\quad - \operatorname{Riem}_{\alpha\beta\gamma}{}^{\mu} h_{\mu\delta} + \frac{1}{2} \operatorname{Riem}_{\alpha\gamma\beta}{}^{\mu} h_{\mu\delta} + \frac{1}{2} \operatorname{Riem}_{\beta\delta\alpha}{}^{\mu} h_{\mu\gamma} \\
&\quad - \frac{1}{2} \operatorname{Riem}_{\alpha\delta\beta}{}^{\mu} h_{\mu\gamma} - \frac{1}{2} \operatorname{Riem}_{\gamma\beta\delta}{}^{\mu} h_{\mu\alpha} \\
&\quad - \frac{1}{2}\nabla_\alpha\nabla_\gamma h_{\beta\delta} - \frac{1}{2}\nabla_\beta\nabla_\delta h_{\alpha\gamma} + \frac{1}{2}\nabla_\alpha\nabla_\delta h_{\beta\gamma} + \frac{1}{2}\nabla_\gamma\nabla_\beta h_{\delta\alpha} \\
&= \frac{1}{2} \operatorname{Riem}_{\alpha\gamma\beta}{}^{\mu} h_{\mu\delta} + \frac{1}{2} \operatorname{Riem}_{\beta\delta\alpha}{}^{\mu} h_{\mu\gamma} \\
&\quad - \frac{1}{2} \operatorname{Riem}_{\alpha\delta\beta}{}^{\mu} h_{\mu\gamma} - \frac{1}{2} \operatorname{Riem}_{\gamma\beta\delta}{}^{\mu} h_{\mu\alpha} \\
&\quad - \frac{1}{2}\nabla_\alpha\nabla_\gamma h_{\beta\delta} - \frac{1}{2}\nabla_\beta\nabla_\delta h_{\alpha\gamma} \\
&\quad + \frac{1}{2}\nabla_\alpha\nabla_\delta h_{\beta\gamma} + \frac{1}{2}\nabla_\gamma\nabla_\beta h_{\delta\alpha}.
\end{aligned} \tag{B.8}$$

Contracting over the second and fourth index, we get after relabelling:

$$\begin{aligned}
\partial_t \operatorname{Ric}_{\alpha\beta} &= (\partial_t g^{\sigma\tau}) \operatorname{Riem}_{\alpha\sigma\beta\tau} + g^{\sigma\tau} (\partial_t \operatorname{Riem}_{\alpha\sigma\beta\tau}) \\
&\overset{(B.3),(B.8)}{=} - \operatorname{Riem}_{\alpha\mu\beta\nu} h^{\mu\nu} + \frac{1}{2} \operatorname{Ric}_{\alpha\mu} h^{\mu}{}_{\beta} + \frac{1}{2} \operatorname{Ric}_{\beta\mu} h^{\mu}{}_{\alpha} \\
&\quad - \frac{1}{2}\nabla_\alpha\nabla_\beta \operatorname{tr}_g h - \frac{1}{2}\Delta h_{\alpha\beta} + \frac{1}{2}\nabla_\alpha \operatorname{div} h_\beta + \frac{1}{2}\nabla_\beta \operatorname{div} h_\alpha.
\end{aligned} \tag{B.9}$$

Contracting again yields:

$$\begin{aligned}
\partial_t \operatorname{Scal} &= \left(\partial_t g^{\alpha\beta}\right) \operatorname{Ric}_{\alpha\beta} + g^{\alpha\beta} (\partial_t \operatorname{Ric}_{\alpha\beta}) \\
&\overset{(B.3),(B.9)}{=} - \operatorname{Ric}_{\alpha\beta} h^{\alpha\beta} - \operatorname{Ric}_{\mu\nu} h^{\mu\nu} + \operatorname{Ric}_{\alpha\mu} h^{\alpha\mu} \\
&\quad - \Delta \operatorname{tr}_g h + \operatorname{div}\operatorname{div} h \\
&= - \operatorname{Ric} : h - \Delta \operatorname{tr}_g h + \operatorname{div}\operatorname{div} h
\end{aligned} \tag{B.10}$$

So we have,

$$\begin{aligned}
\partial_t \mathring{\operatorname{Ric}}_{\alpha\beta} &= \partial_t \operatorname{Ric}_{\alpha\beta} - \frac{1}{n+1} (\partial_t \operatorname{Scal}) g_{\alpha\beta} - \frac{1}{n+1} \operatorname{Scal} (\partial_t g_{\alpha\beta}) \\
&\overset{(B.2),(B.9),(B.10)}{=} - \operatorname{Riem}_{\alpha\mu\beta\nu} h^{\mu\nu} + \frac{1}{2} \operatorname{Ric}_{\alpha\mu} h^{\mu}{}_{\beta} + \frac{1}{2} \operatorname{Ric}_{\beta\mu} h^{\mu}{}_{\alpha} \\
&\quad + \frac{1}{n+1} \operatorname{Ric} : h g_{\alpha\beta} - \frac{1}{n+1} \operatorname{Scal} h_{\alpha\beta} \\
&\quad - \frac{1}{2}\nabla_\alpha\nabla_\beta \operatorname{tr}_g h - \frac{1}{2}\Delta h_{\alpha\beta} + \frac{1}{n+1}\Delta \operatorname{tr}_g h g_{\alpha\beta} \\
&\quad + \frac{1}{2}\nabla_\alpha \operatorname{div} h_\beta + \frac{1}{2}\nabla_\beta \operatorname{div} h_\alpha - \frac{1}{n+1} \operatorname{div}\operatorname{div} h g_{\alpha\beta}.
\end{aligned} \tag{B.11}$$

On the other hand,

$$\begin{aligned}
\partial_t \|\mathrm{Ric}\|_g^2 &= 2\,\mathrm{Ric}_{\mu\nu}\, g^{\mu\alpha} \left(\partial_t g^{\nu\beta}\right) \mathrm{Ric}_{\alpha\beta} + 2\,\mathrm{Ric}_{\mu\nu}\, g^{\mu\alpha} g^{\nu\beta} \left(\partial_t \mathrm{Ric}_{\alpha\beta}\right) \\
&\overset{(B.3),(B.9)}{=} -2\,\mathrm{Ric}_{\nu\mu}\, \mathrm{Ric}^{\mu}{}_{\beta}\, h^{\nu\beta} - 2\,\mathrm{Riem}_{\alpha\mu\beta\nu}\, \mathrm{Ric}^{\alpha\beta}\, h^{\mu\nu} + \mathrm{Ric}_{\beta}{}^{\alpha}\, \mathrm{Ric}_{\alpha\mu}\, h^{\beta\mu} \\
&\quad + \mathrm{Ric}_{\alpha}{}^{\beta}\, \mathrm{Ric}_{\beta\mu}\, h^{\alpha\mu} - \mathrm{Ric}^{\alpha\beta}\, \nabla_\alpha \nabla_\beta \operatorname{tr}_g h - \mathrm{Ric}^{\alpha\beta}\, \Delta h_{\alpha\beta} \\
&\quad + \mathrm{Ric}^{\alpha\beta}\, \nabla_\alpha \operatorname{div} h_\beta + \mathrm{Ric}^{\alpha\beta}\, \nabla_\beta \operatorname{div} h_\alpha \\
&= -2\,\mathrm{Riem}_{\alpha\mu\beta\nu}\, \mathrm{Ric}^{\alpha\beta}\, h^{\mu\nu} \\
&\quad - \mathrm{Ric} : \nabla^2 \operatorname{tr}_g h - \mathrm{Ric} : \Delta h + 2\,\mathrm{Ric} : \nabla \operatorname{div} h, \qquad \text{(B.12)}
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \mathrm{Scal}^2 &= 2\,\mathrm{Scal}\left(\partial_t \mathrm{Scal}\right) \\
&\overset{(B.10)}{=} -2\,\mathrm{Scal}\,\mathrm{Ric} : h - 2\,\mathrm{Scal}\,\Delta \operatorname{tr}_g h + 2\,\mathrm{Scal}\operatorname{div}\operatorname{div} h, \qquad \text{(B.13)}
\end{aligned}$$

so that

$$\begin{aligned}
\partial_t \|\mathring{\mathrm{Ric}}\|_g^2 &= \partial_t \left\| \mathrm{Ric} - \frac{1}{n+1}\,\mathrm{Scal}\, g \right\|_g^2 = \partial_t \left(\|\mathrm{Ric}\|_g^2 - \frac{1}{n+1}\,\mathrm{Scal}^2 \right) \\
&\overset{(B.12),(B.13)}{=} -2\,\mathrm{Riem}_{\alpha\mu\beta\nu}\, \mathrm{Ric}^{\alpha\beta}\, h^{\mu\nu} + 2\frac{1}{n+1}\,\mathrm{Scal}\,\mathrm{Riem}_{\alpha\mu\beta\nu}\, g^{\alpha\beta} h^{\mu\nu} \\
&\quad - \mathrm{Ric} : \nabla^2 \operatorname{tr}_g h - \mathrm{Ric} : \Delta h + 2\frac{1}{n+1}\,\mathrm{Scal}\,\Delta \operatorname{tr}_g h \\
&\quad + 2\,\mathrm{Ric} : \nabla \operatorname{div} h - 2\frac{1}{n+1}\,\mathrm{Scal}\operatorname{div}\operatorname{div} h \\
&= -2\,\mathrm{Riem}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\, \mathring{\mathrm{Ric}}_{\alpha\beta} h^{\mu\nu} \\
&\quad - \mathring{\mathrm{Ric}} : \nabla^2 \operatorname{tr}_g h - \mathring{\mathrm{Ric}} : \Delta h + 2\mathring{\mathrm{Ric}} : \nabla \operatorname{div} h. \qquad \text{(B.14)}
\end{aligned}$$

As a result, omitting the volume element for simplicity,

$$\begin{aligned}
\partial_t \int_M \|\mathring{\mathrm{Ric}}\|_g^2 &\overset{(B.4)}{=} \frac{1}{2} \int_M \operatorname{tr}_g h\, \|\mathring{\mathrm{Ric}}\|_g^2 + \int_M \partial_t \|\mathring{\mathrm{Ric}}\|_g^2 \\
&\overset{(B.14)}{=} \frac{1}{2} \int_M \operatorname{tr}_g h\, \|\mathring{\mathrm{Ric}}\|_g^2 - 2 \int_M \mathrm{Riem}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\, \mathring{\mathrm{Ric}}_{\alpha\beta} h^{\mu\nu} \\
&\quad - \int_M \mathring{\mathrm{Ric}} : \nabla^2 \operatorname{tr}_g h - \int_M \mathring{\mathrm{Ric}} : \Delta h + 2 \int_M \mathring{\mathrm{Ric}} : \nabla \operatorname{div} h. \qquad \text{(B.15)}
\end{aligned}$$

Also, we find that

$$
\begin{aligned}
\partial_t \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right) &= \partial_t \left(\mathrm{Scal} - \frac{1}{\mathrm{Vol}} \int_M \mathrm{Scal}\right) \\
&\overset{(B.4),(B.5)}{=} \partial_t \,\mathrm{Scal} - \frac{1}{2\mathrm{Vol}} \int_M \mathrm{tr}_g\, h \,\mathrm{Scal} - \frac{1}{\mathrm{Vol}} \int_M \partial_t \,\mathrm{Scal} \\
&\overset{(B.10)}{=} - \mathrm{Ric} : h - \Delta \,\mathrm{tr}_g\, h + \mathrm{div}\,\mathrm{div}\, h - \frac{1}{2\mathrm{Vol}} \int_M \mathrm{tr}_g\, h \,\mathrm{Scal} \\
&\qquad + \frac{1}{\mathrm{Vol}} \int_M \mathrm{Ric} : h + \frac{1}{\mathrm{Vol}} \int_M \Delta \,\mathrm{tr}_g\, h - \frac{1}{\mathrm{Vol}} \int_M \mathrm{div}\,\mathrm{div}\, h, \qquad \text{(B.16)}
\end{aligned}
$$

yielding:

$$
\begin{aligned}
\partial_t \int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)^2 &\overset{(B.4)}{=} \frac{1}{2} \int_M \mathrm{tr}_g\, h \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)^2 \\
&\quad + 2 \int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right) \left(\partial_t \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)\right) \\
&\overset{(B.16)}{=} \frac{1}{2} \int_M \mathrm{tr}_g\, h \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)^2 - 2 \int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right) (\mathrm{Ric} : h) \\
&\quad - 2 \int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right) (\Delta \,\mathrm{tr}_g\, h) \\
&\quad + 2 \int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right) (\mathrm{div}\,\mathrm{div}\, h)\,, \qquad \text{(B.17)}
\end{aligned}
$$

since $\int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right) = 0$.

Now, at $t = 0$, $(M, g)\big|_{t=0}$ is a subset of the standard $(n+1)$–sphere with radius R, and we have

$$
\begin{aligned}
g_{\alpha\beta}\big|_{t=0} &= \eta_{\alpha\beta}, \\
\mathrm{Riem}_{\alpha\beta\gamma\delta}\big|_{t=0} &= \frac{1}{R^2} \left(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}\right), \\
\mathrm{Riem}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\big|_{t=0} &= \frac{1}{R^2} \eta^{\alpha\beta}\eta_{\mu\nu} - \frac{1}{R^2} \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu}, \\
\mathrm{Ric}_{\alpha\beta}\big|_{t=0} &= \frac{n}{R^2} \eta_{\alpha\beta}, \\
\mathrm{Scal}\big|_{t=0} &= \frac{n(n+1)}{R^2}, \\
\mathring{\mathrm{Ric}}_{\alpha\beta}\big|_{t=0} &= 0.
\end{aligned}
$$

Considering the two quantities $\int_M \|\mathring{\mathrm{Ric}}\|_g^2$ and $\int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)^2$, the evolutions of which are given by (B.15) and (B.17), respectively, we see that both vanish at $t = 0$. Moreover, since both are non–negative at all times, we infer from their evolutions that they must be minimal initially. We are thus interested in their second variations at $t = 0$. For this, we calculate:

$$\begin{aligned}
\partial_t\Big|_{t=0}\mathring{\mathrm{Ric}}_{\alpha\beta} &\overset{(B.11)}{=} -\frac{1}{R^2}\operatorname{tr}_\eta h\eta_{\alpha\beta} + \frac{1}{R^2}h_{\alpha\beta} + \frac{n}{R^2}h_{\alpha\beta} + \frac{n}{(n+1)R^2}\operatorname{tr}_\eta h\eta_{\alpha\beta} - \frac{n}{R^2}h_{\alpha\beta}\\
&\quad - \frac{1}{2}\nabla_\alpha\nabla_\beta\operatorname{tr}_\eta h - \frac{1}{2}\Delta h_{\alpha\beta} + \frac{1}{n+1}\Delta\operatorname{tr}_\eta h\eta_{\alpha\beta}\\
&\quad + \frac{1}{2}\nabla_\alpha\operatorname{div}h_\beta + \frac{1}{2}\nabla_\beta\operatorname{div}h_\alpha - \frac{1}{n+1}\operatorname{div}\operatorname{div}h\eta_{\alpha\beta}\\
&= \frac{1}{R^2}\mathring{h}_{\alpha\beta} - \frac{1}{2}\nabla_\alpha\nabla_\beta\operatorname{tr}_\eta h - \frac{1}{2}\Delta h_{\alpha\beta} + \frac{1}{n+1}\Delta\operatorname{tr}_\eta h\eta_{\alpha\beta}\\
&\quad + \frac{1}{2}\nabla_\alpha\operatorname{div}h_\beta + \frac{1}{2}\nabla_\beta\operatorname{div}h_\alpha - \frac{1}{n+1}\operatorname{div}\operatorname{div}h\eta_{\alpha\beta},
\end{aligned}\tag{B.18}$$

where, for all t, $\mathring{h} = h - \frac{1}{n+1}\operatorname{tr}_g hg$. Then, in view of the fact that $\mathring{\mathrm{Ric}}$ vanishes at $t = 0$,

$$\begin{aligned}
\partial_t^2\Big|_{t=0}\int_M \|\mathring{\mathrm{Ric}}\|_g^2 &\overset{(B.15)}{=} -2\int_M \left(\mathrm{Riem}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\right)\Big|_{t=0}\left(\partial_t\Big|_{t=0}\mathring{\mathrm{Ric}}_{\alpha\beta}\right)h^{\mu\nu}\\
&\quad - \int_M\left(\partial_t\Big|_{t=0}\mathring{\mathrm{Ric}}_{\alpha\beta}\right)\nabla^\alpha\nabla^\beta\operatorname{tr}_\eta h - \int_M\left(\partial_t\Big|_{t=0}\mathring{\mathrm{Ric}}_{\alpha\beta}\right)\Delta h^{\alpha\beta}\\
&\quad + 2\int_M\left(\partial_t\Big|_{t=0}\mathring{\mathrm{Ric}}_{\alpha\beta}\right)\nabla^\alpha\operatorname{div}h^\beta
\end{aligned}$$

$$\begin{aligned}
&\overset{(B.18)}{=} 0 + \frac{2}{R^4}\int_M h{:}\mathring{h} - \frac{1}{R^2}\int_M h{:}\nabla^2\operatorname{tr}_\eta h - \frac{1}{R^2}\int_M h{:}\Delta h + \frac{2}{(n+1)R^2}\int_M \operatorname{tr}_\eta h\,(\Delta\operatorname{tr}_\eta h)\\
&\quad + \frac{2}{R^2}\int_M h{:}\nabla\operatorname{div}h - \frac{2}{(n+1)R^2}\int_M\operatorname{tr}_\eta h\,(\operatorname{div}\operatorname{div}h)\\
&\quad - \frac{1}{R^2}\int_M\mathring{h}{:}\nabla^2\operatorname{tr}_\eta h + \frac{1}{2}\int_M\left\|\nabla^2\operatorname{tr}_\eta h\right\|_\eta^2 + \frac{1}{2}\int_M\Delta h{:}\nabla^2\operatorname{tr}_\eta h\\
&\quad - \frac{1}{n+1}\int_M(\Delta\operatorname{tr}_\eta h)^2 - \int_M\nabla^2\operatorname{tr}_\eta h{:}\nabla\operatorname{div}h + \frac{1}{n+1}\int_M(\operatorname{div}\operatorname{div}h)\,(\Delta\operatorname{tr}_\eta h)\\
&\quad - \frac{1}{R^2}\int_M\mathring{h}{:}\Delta h + \frac{1}{2}\int\Delta h{:}\nabla^2\operatorname{tr}_\eta h + \frac{1}{2}\int_M\|\Delta h\|_\eta^2\\
&\quad - \frac{1}{n+1}\int_M(\Delta\operatorname{tr}_\eta h)^2 - \int_M\Delta h{:}\nabla\operatorname{div}h + \frac{1}{n+1}\int_M(\operatorname{div}\operatorname{div}h)\,(\Delta\operatorname{tr}_\eta h)\\
&\quad + \frac{2}{R^2}\int_M\mathring{h}{:}\nabla\operatorname{div}h - \int_M\nabla^2\operatorname{tr}_\eta h{:}\nabla\operatorname{div}h - \int_M\Delta h{:}\nabla\operatorname{div}h\\
&\quad + \frac{2}{n+1}\int_M(\Delta\operatorname{tr}_\eta h)\,(\operatorname{div}\operatorname{div}h) + \int_M\|\nabla\operatorname{div}h\|_\eta^2\\
&\quad + \int_M\nabla^\beta\operatorname{div}h^\alpha\nabla_\alpha\operatorname{div}h_\beta - \frac{2}{n+1}\int_M(\operatorname{div}\operatorname{div}h)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{R^4}\int_M \mathring{h}{:}h - \frac{2}{R^2}\int_M \mathring{h}{:}\nabla^2 \operatorname{tr}_\eta h \\
&\quad - \frac{2}{R^2}\int_M \mathring{h}{:}\Delta h + \frac{4}{R^2}\int_M \mathring{h}{:}\nabla \operatorname{div} h \\
&\quad + \frac{1}{2}\int_M \left\|\nabla^2 \operatorname{tr}_\eta h\right\|_\eta^2 + \frac{1}{2}\int_M \|\Delta h\|_\eta^2 + \int_M \|\nabla \operatorname{div} h\|_\eta^2 \\
&\quad - \frac{2}{n+1}\int_M (\Delta \operatorname{tr}_\eta h)^2 - \frac{2}{n+1}\int_M (\operatorname{div}\operatorname{div} h)^2 \\
&\quad + \int_M \nabla^2 \operatorname{tr}_\eta h{:}\Delta h - 2\int_M \nabla^2 \operatorname{tr}_\eta h{:}\nabla \operatorname{div} h \\
&\quad - 2\int_M \Delta h{:}\nabla \operatorname{div} h + \int_M \nabla^\beta \operatorname{div} h^\alpha \nabla_\alpha \operatorname{div} h_\beta \\
&\quad + \frac{4}{n+1}\int_M (\Delta \operatorname{tr}_g h)(\operatorname{div}\operatorname{div} h)\,.
\end{aligned}
\tag{B.19}
$$

Similarly, we calculate

$$
\begin{aligned}
\partial_t\big|_{t=0}\left(\operatorname{Scal} - \overline{\operatorname{Scal}}\right) &\overset{(B.16)}{=} -\Delta \operatorname{tr}_\eta h + \operatorname{div}\operatorname{div} h - \frac{n}{R^2}\operatorname{tr}_\eta h - \frac{n(n+1)}{2R^2\mathrm{Vol}} \underbrace{\int_M \operatorname{tr}_\eta h}_{\overset{(B.1)}{=} 0} \\
&\quad + \frac{1}{\mathrm{Vol}}\int_M \Delta \operatorname{tr}_\eta h - \frac{1}{\mathrm{Vol}}\int_M \operatorname{div}\operatorname{div} h + \frac{n}{R^2\mathrm{Vol}} \underbrace{\int_M \operatorname{tr}_\eta h}_{\overset{(B.1)}{=} 0} \\
&= -\Delta \operatorname{tr}_\eta h + \operatorname{div}\operatorname{div} h - \frac{n}{R^2}\operatorname{tr}_\eta h + \frac{1}{\mathrm{Vol}}\int_M \Delta \operatorname{tr}_\eta h \\
&\quad - \frac{1}{\mathrm{Vol}}\int_M \operatorname{div}\operatorname{div} h,
\end{aligned}
\tag{B.20}
$$

so that (since $\operatorname{Scal}\big|_{t=0}$ is constant on M)

$$
\begin{aligned}
\partial_t^2\big|_{t=0}\int_M \left(\operatorname{Scal} - \overline{\operatorname{Scal}}\right)^2 &\overset{(B.17)}{=} -\frac{2n}{R^2}\int_M \left(\partial_t\big|_{t=0}\left(\operatorname{Scal} - \overline{\operatorname{Scal}}\right)\right)\operatorname{tr}_\eta h \\
&\quad - 2\int_M \left(\partial_t\big|_{t=0}\left(\operatorname{Scal} - \overline{\operatorname{Scal}}\right)\right)\Delta \operatorname{tr}_\eta h \\
&\quad + 2\int_M \left(\partial_t\big|_{t=0}\left(\operatorname{Scal} - \overline{\operatorname{Scal}}\right)\right)\operatorname{div}\operatorname{div} h
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{(B.20)}}{=} \frac{2n}{R^2}\int_M \operatorname{tr}_\eta h \Delta \operatorname{tr}_\eta h - \frac{2n}{R^2}\int_M \operatorname{tr}_\eta h \operatorname{div}\operatorname{div} h \\
&\quad + \frac{2n^2}{R^4}\int_M (\operatorname{tr}_\eta h)^2 - \frac{2n}{R^2 \mathrm{Vol}}\left(\int_M \Delta \operatorname{tr}_\eta h\right)\underbrace{\left(\int_M \operatorname{tr}_\eta h\right)}_{\overset{\text{(B.1)}}{=}0} \\
&\quad + \frac{2n}{R^2 \mathrm{Vol}}\left(\int_M \operatorname{div}\operatorname{div} h\right)\underbrace{\left(\int_M \operatorname{tr}_\eta h\right)}_{\overset{\text{(B.1)}}{=}0} \\
&\quad + 2\int_M (\Delta \operatorname{tr}_\eta h)^2 - 2\int_M \Delta \operatorname{tr}_\eta h \operatorname{div}\operatorname{div} h + \frac{2n}{R^2}\int_M \operatorname{tr}_\eta h \Delta \operatorname{tr}_\eta h \\
&\quad - \frac{2}{\mathrm{Vol}}\left(\int_M \Delta \operatorname{tr}_\eta h\right)^2 + \frac{2}{\mathrm{Vol}}\left(\int_M \Delta \operatorname{tr}_\eta h\right)\left(\int_M \operatorname{div}\operatorname{div} h\right) \\
&\quad - 2\int_M \Delta \operatorname{tr}_\eta h \operatorname{div}\operatorname{div} h + 2\int_M (\operatorname{div}\operatorname{div} h)^2 - \frac{2n}{R^2}\int_M \operatorname{tr}_\eta h \operatorname{div}\operatorname{div} h \\
&\quad + \frac{2}{\mathrm{Vol}}\left(\int_M \Delta \operatorname{tr}_\eta h\right)\left(\int_M \operatorname{div}\operatorname{div} h\right) - \frac{2}{\mathrm{Vol}}\left(\int_M \operatorname{div}\operatorname{div} h\right)^2 \\
&= \frac{2n^2}{R^4}\int_M (\operatorname{tr}_\eta h)^2 + \frac{4n}{R^2}\int_M \operatorname{tr}_\eta h \Delta \operatorname{tr}_\eta h \\
&\quad - \frac{4n}{R^2}\int_M \operatorname{tr}_\eta h \operatorname{div}\operatorname{div} h + 2\int_M (\Delta \operatorname{tr}_\eta h)^2 \\
&\quad - \frac{2}{\mathrm{Vol}}\left(\int_M \Delta \operatorname{tr}_\eta h\right)^2 + 2\int_M (\operatorname{div}\operatorname{div} h)^2 \\
&\quad - \frac{2}{\mathrm{Vol}}\left(\int_M \operatorname{div}\operatorname{div} h\right)^2 - 4\int \Delta \operatorname{tr}_\eta h \operatorname{div}\operatorname{div} h \\
&\quad + \frac{4}{\mathrm{Vol}}\left(\int_M \Delta \operatorname{tr}_\eta h\right)\left(\int_M \operatorname{div}\operatorname{div} h\right)
\end{aligned}
\tag{B.21}
$$

3. The first and second variations of the quantities on ∂M

We shall now focus our attention to the boundary ∂M of M. Its induced metric $\widetilde{g}$ is given by

$$\widetilde{g}_{ij} = g_{ij}.$$

Also, the inverse of $\widetilde{g}$ is quickly computed to be

$$\left(\widetilde{g}^{-1}\right)^{ij} = g^{ij} - \nu^i \nu^j,$$

where ν denotes the outer unit normal field on ∂M given in our coordinates by

$$\nu^\alpha = -\frac{g^{0\alpha}}{\sqrt{g^{00}}}.$$

Notice that, for any two–tensor field B on M, its restriction to ∂M satisfies

$$\mathrm{tr}_{\widetilde{g}} B = \left(\widetilde{g}^{-1}\right)^{ij} B_{ij} = g^{ij} B_{ij} - \nu^i \nu^j B_{ij} = g^{\sigma\tau} B_{\sigma\tau} - \nu^\sigma \nu^\tau B_{\sigma\tau} = \mathrm{tr}_g B - B(\nu,\nu).$$

We start, again, by the evolution of the metric:

$$\partial_t \widetilde{g}_{ij} \overset{(B.2)}{=} h_{ij}. \tag{B.22}$$

Then, similarly to the computation of (B.4),

$$\partial_t \sqrt{\det \widetilde{g}} = \frac{\sqrt{\det \widetilde{g}}}{2} \mathrm{tr}_{\widetilde{g}} (\partial_t \widetilde{g}) \overset{(B.22)}{=} \frac{1}{2} \left(\mathrm{tr}_g h - h(\nu,\nu)\right) \sqrt{\det \widetilde{g}}, \tag{B.23}$$

Also, the outer unit normal evolves according to

$$\begin{aligned}
\partial_t \nu^\alpha &= \frac{g^{0\alpha}}{2\left(g^{00}\right)^{3/2}} \left(\partial_t g^{00}\right) - \frac{1}{\sqrt{g^{00}}} \left(\partial_t g^{0\alpha}\right) \\
&\overset{(B.3)}{=} \frac{g^{0\alpha}}{2\left(g^{00}\right)^{3/2}} \left(-g^{0\sigma} g^{0\tau} h_{\sigma\tau}\right) - \frac{1}{\sqrt{g^{00}}} \left(-g^{0\sigma} g^{\alpha\tau} h_{\sigma\tau}\right) = \frac{1}{2} \nu^\alpha \nu^\sigma \nu^\tau h_{\sigma\tau} - \nu^\sigma h^\alpha{}_\sigma \\
&= \frac{1}{2} h(\nu,\nu) \nu^\alpha - \left(h^\sharp(\nu)\right)^\alpha,
\end{aligned} \tag{B.24}$$

where $\left(h^\sharp(\nu)\right)^\alpha = h^\alpha{}_\sigma \nu^\sigma$. Consequently,

$$\begin{aligned}
\partial_t \left(\widetilde{g}^{-1}\right)^{ij} &= \partial_t g^{ij} - \left(\partial_t \nu^i\right) \nu^j - \nu^i \left(\partial_t \nu^j\right) \\
&\overset{(B.3),(B.24)}{=} -h^{ij} - h(\nu,\nu) \nu^i \nu^j + \left(h^\sharp(\nu)\right)^i \nu^j + \nu^i \left(h^\sharp(\nu)\right)^j
\end{aligned} \tag{B.25}$$

On the other hand, similarly to the computation of (B.3), we have

$$\partial_t \left(\widetilde{g}^{-1}\right)^{ij} = -\left(\widetilde{g}^{-1}\right)^{ik} \left(\widetilde{g}^{-1}\right)^{jl} h_{kl} \tag{B.26}$$

Now, denoting by ∂_α the coordinate vector fields, we get for the evolution of the second fundamental form A:

$$\begin{aligned}
\partial_t A_{ij} &= \partial_t \left(-g\left(\nabla_i \partial_j, \nu\right)\right) = \partial_t \left(-\Gamma^\sigma_{ij} g\left(\partial_\sigma, \nu\right)\right) = \partial_t \left(\Gamma^\sigma_{ij} \frac{\delta^0{}_\sigma}{\sqrt{g^{00}}}\right) \\
&= \partial_t \left(\frac{\Gamma^0_{ij}}{\sqrt{g^{00}}}\right) = -\frac{\Gamma^0_{ij}}{2\left(g^{00}\right)^{3/2}} \left(\partial_t g^{00}\right) + \frac{1}{\sqrt{g^{00}}} \partial_t \Gamma^0_{ij} \\
&\overset{(B.3),(B.6)}{=} \frac{1}{2} \Gamma^0_{ij} \frac{h^{00}}{\left(g^{00}\right)^{3/2}} + \frac{g^{0\tau}}{2\sqrt{g^{00}}} \left(\nabla_i h_{j\tau} + \nabla_j h_{i\tau} - \nabla_\tau h_{ij}\right) \\
&= \frac{1}{2} A_{ij} \frac{h^{00}}{g^{00}} - \frac{1}{2} \left(\left(\nabla h\right)(\nu)\right)_{ij} - \frac{1}{2} \left(\left(\nabla h\right)(\nu)\right)_{ji} + \frac{1}{2} \left(\nabla_\nu h\right)_{ij} \\
&= \frac{1}{2} A_{ij} h(\nu,\nu) - \frac{1}{2} \left(\left(\nabla h\right)(\nu)\right)_{ij} - \frac{1}{2} \left(\left(\nabla h\right)(\nu)\right)_{ji} + \frac{1}{2} \left(\nabla_\nu h\right)_{ij},
\end{aligned} \tag{B.27}$$

where $\left((\nabla h)(\nu)\right)_{ij} = \nabla_i h_{\sigma j}\nu^\sigma$. Therefore,

$$
\begin{aligned}
\partial_t H &= \left(\partial_t \left(\widetilde{g}^{-1}\right)^{ij}\right) A_{ij} + \left(\widetilde{g}^{-1}\right)^{ij} \partial_t A_{ij} \\
&\overset{(B.26),(B.27)}{=} -h\,\widetilde{:}\,A + \frac{1}{2}h(\nu,\nu)H - \mathrm{tr}_{\widetilde{g}}\left((\nabla h)(\nu)\right) + \frac{1}{2}\mathrm{tr}_{\widetilde{g}}\left(\nabla_\nu h\right) \\
&= -A\,\widetilde{:}\,h + \frac{1}{2}h(\nu,\nu)H - \mathrm{tr}_g\left((\nabla h)(\nu)\right) + \left(\nabla_\nu h\right)(\nu,\nu) \\
&\quad + \frac{1}{2}\mathrm{tr}_g\left(\nabla_\nu h\right) - \frac{1}{2}\left(\nabla_\nu h\right)(\nu,\nu) \\
&= -A\,\widetilde{:}\,h + \frac{1}{2}h(\nu,\nu)H - \left(\operatorname{div} h\right)(\nu) + \frac{1}{2}\left(\nabla_\nu h\right)(\nu,\nu) + \frac{1}{2}\nabla_\nu \mathrm{tr}_g\, h. \qquad \text{(B.28)}
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
\partial_t \mathring{A}_{ij} &= \partial_t A_{ij} - \frac{1}{n}\left(\partial_t H\right)\widetilde{g}_{ij} - \frac{1}{n}H\left(\partial_t \widetilde{g}_{ij}\right) \\
&\overset{(B.22),(B.27),(B.28)}{=} \frac{1}{2}h(\nu,\nu)\mathring{A}_{ij} + \frac{1}{n}A\,\widetilde{:}\,h g_{ij} - \frac{1}{n}H h_{ij} \\
&\quad - \frac{1}{2}\left((\nabla h)(\nu)\right)_{ij} - \frac{1}{2}\left((\nabla h)(\nu)\right)_{ji} + \frac{1}{n}\operatorname{div} h(\nu) g_{ij} \\
&\quad + \frac{1}{2}\left(\nabla_\nu h\right)_{ij} - \frac{1}{2n}\nabla_\nu \mathrm{tr}_g\, h g_{ij} - \frac{1}{2n}\left(\nabla_\nu h\right)(\nu,\nu) g_{ij}. \qquad \text{(B.29)}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\partial_t \|\mathring{A}\|^2_{\widetilde{g}} &= \partial_t\left(\|A\|^2_{\widetilde{g}} - \frac{1}{n}H^2\right) \\
&= 2A_{ik}\left(\widetilde{g}^{-1}\right)^{kl} A_{jl}\left(\partial_t\left(\widetilde{g}^{-1}\right)^{ij}\right) + 2A\,\widetilde{:}\left(\partial_t A\right) - \frac{2}{n}H\left(\partial_t H\right) \\
&\overset{(B.26),(B.27),(B.28)}{=} -2A^2\,\widetilde{:}\,h + \|A\|^2_{\widetilde{g}}h(\nu,\nu) - 2A\,\widetilde{:}\left((\nabla h)(\nu)\right) + A\,\widetilde{:}\left(\nabla_\nu h\right) \\
&\quad + \frac{2}{n}HA\,\widetilde{:}\,h - \frac{1}{n}h(\nu,\nu)H^2 + \frac{2}{n}H\operatorname{div} h(\nu) \\
&\quad - \frac{1}{n}H\left(\nabla_\nu h\right)(\nu,\nu) - \frac{1}{n}H\nabla_\nu \mathrm{tr}_g\, h \\
&= -2\left(A\mathring{A}\right)\widetilde{:}\,h + \|\mathring{A}\|^2_{\widetilde{g}}h(\nu,\nu) \\
&\quad - 2\mathring{A}\,\widetilde{:}\left((\nabla h)(\nu)\right) + \mathring{A}\,\widetilde{:}\left(\nabla_\nu h\right), \qquad \text{(B.30)}
\end{aligned}
$$

where, for any two–tensor fields $\widetilde{B}$ and $\widetilde{C}$ on ∂M, $\widetilde{B}\widetilde{C}$ is defined as $(\widetilde{B}\widetilde{C})_{ij} = \widetilde{B}_{ik}\left(\widetilde{g}^{-1}\right)^{kl}\widetilde{C}_{lj}$, and $\widetilde{B}^2 = \widetilde{B}\widetilde{B}$. Hence,

$$
\begin{aligned}
\partial_t \int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} &\overset{(B.23)}{=} \frac{1}{2}\int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} \operatorname{tr}_g h - \frac{1}{2}\int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} h(\nu,\nu) + \int_{\partial M} \partial_t \|\mathring{A}\|^2_{\widetilde{g}} \\
&\overset{(B.30)}{=} \frac{1}{2}\int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} \operatorname{tr}_g h - \frac{1}{2}\int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} h(\nu,\nu) - 2\int_{\partial M} (A\mathring{A})\,\widetilde{:}\,h \\
&\quad + \int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} h(\nu,\nu) - 2\int_{\partial M} \mathring{A}\,\widetilde{:}\,((\nabla h)(\nu)) + \int_{\partial M} \mathring{A}\,\widetilde{:}\,(\nabla_\nu h) \\
&= \frac{1}{2}\int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} \operatorname{tr}_g h + \frac{1}{2}\int_{\partial M} \|\mathring{A}\|^2_{\widetilde{g}} h(\nu,\nu) \\
&\quad - 2\int_{\partial M} (A\mathring{A})\,\widetilde{:}\,h - 2\int_{\partial M} \mathring{A}\,\widetilde{:}\,((\nabla h)(\nu)) + \int_{\partial M} \mathring{A}\,\widetilde{:}\,(\nabla_\nu h) \qquad \text{(B.31)}
\end{aligned}
$$

Also, since for any smooth function f on ∂M,

(B.32)

$$
\begin{aligned}
\partial_t \overline{f} &= \partial_t ⨍_{\partial M} f = \partial_t \left(\frac{1}{\widetilde{\operatorname{Vol}}} \int_{\partial M} f \right) \\
&= -\frac{1}{\widetilde{\operatorname{Vol}}^2}\left(\partial_t \int_{\partial M} 1\right)\int_{\partial M} f + \frac{1}{\widetilde{\operatorname{Vol}}}\partial_t \int_{\partial M} f \\
&\overset{(B.23)}{=} -\frac{1}{2} ⨍_{\partial M} (\operatorname{tr}_g h - h(\nu,\nu)) ⨍_{\partial M} f + \frac{1}{2} ⨍_{\partial M} (\operatorname{tr}_g h - h(\nu,\nu)) f + ⨍_{\partial M} \partial_t f \\
&= \frac{1}{2} ⨍_{\partial M} \left(f - \overline{f}\right)(\operatorname{tr}_g h - h(\nu,\nu)) + ⨍_{\partial M} \partial_t f, \qquad \text{(B.33)}
\end{aligned}
$$

we have

$$
\begin{aligned}
\partial_t \left(H - \overline{H}\right) &\overset{(B.33)}{=} \partial_t H - ⨍_{\partial M} \partial_t H - \frac{1}{2} ⨍_{\partial M} \left(H - \overline{H}\right)(\operatorname{tr}_g h - h(\nu,\nu)) \\
&\overset{(B.28)}{=} -A\,\widetilde{:}\,h + ⨍_{\partial M} A\,\widetilde{:}\,h + \frac{1}{2}h(\nu,\nu)H - \frac{1}{2} ⨍_{\partial M} h(\nu,\nu)H \\
&\quad - \operatorname{div} h(\nu) + ⨍_{\partial M} \operatorname{div} h(\nu) + \frac{1}{2}(\nabla_\nu h)(\nu,\nu) - \frac{1}{2} ⨍_{\partial M} (\nabla_\nu h)(\nu,\nu) \\
&\quad + \frac{1}{2}\nabla_\nu \operatorname{tr}_g h - \frac{1}{2} ⨍_{\partial M} \nabla_\nu \operatorname{tr}_g h \\
&\quad - \frac{1}{2} ⨍_{\partial M} \left(H - \overline{H}\right)\operatorname{tr}_g h + \frac{1}{2} ⨍_{\partial M} \left(H - \overline{H}\right) h(\nu,\nu). \qquad \text{(B.34)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\partial_t \int_{\partial M} \left(H-\overline{H}\right)^2 &\overset{(B.23)}{=} \frac{1}{2}\int_{\partial M} \left(H-\overline{H}\right)^2 \operatorname{tr}_g h - \frac{1}{2}\int_{\partial M} \left(H-\overline{H}\right)^2 h(\nu,\nu) \\
&\quad + 2\int_{\partial M} \left(H-\overline{H}\right) \partial_t \left(H-\overline{H}\right) \\
&\overset{(B.34)}{=} \frac{1}{2}\int_{\partial M} \left(H-\overline{H}\right)^2 \operatorname{tr}_g h - \frac{1}{2}\int_{\partial M} \left(H-\overline{H}\right)^2 h(\nu,\nu) \\
&\quad - 2\int_{\partial M} \left(H-\overline{H}\right) A\tilde{:}h + \int_{\partial M} \left(H-\overline{H}\right) H h(\nu,\nu) \\
&\quad - 2\int_{\partial M} \left(H-\overline{H}\right) (\operatorname{div} h)(\nu) + \int_{\partial M} \left(H-\overline{H}\right) (\nabla_\nu h)(\nu,\nu) \\
&\quad + \int_{\partial M} \left(H-\overline{H}\right) \nabla_\nu \operatorname{tr}_g h \\
&= \frac{1}{2}\int_{\partial M} \left(H-\overline{H}\right)^2 \operatorname{tr}_g h + \frac{1}{2}\int_{\partial M} \left(H-\overline{H}\right)^2 h(\nu,\nu) \\
&\quad - 2\int_{\partial M} \left(H-\overline{H}\right) A\tilde{:}h + \overline{H}\int_{\partial M} \left(H-\overline{H}\right) h(\nu,\nu) \\
&\quad - 2\int_{\partial M} \left(H-\overline{H}\right) (\operatorname{div} h)(\nu) + \int_{\partial M} \left(H-\overline{H}\right) (\nabla_\nu h)(\nu,\nu) \\
&\quad + \int_{\partial M} \left(H-\overline{H}\right) \nabla_\nu \operatorname{tr}_g h, \qquad \text{(B.35)}
\end{aligned}
$$

where we have used that $\int_{\partial M} \left(H-\overline{H}\right) = 0$.
Finally, just in the same way that we established (B.15), we have

$$
\begin{aligned}
\partial_t \int_{\partial M} \|\mathring{\operatorname{Ric}}\|_g^2 &\overset{(B.23)}{=} \frac{1}{2}\int_{\partial M} \operatorname{tr}_g h \, \|\mathring{\operatorname{Ric}}\|_g^2 - \frac{1}{2}\int_{\partial M} h(\nu,\nu)\, \|\mathring{\operatorname{Ric}}\|_g^2 + \int_{\partial M} \partial_t \|\mathring{\operatorname{Ric}}\|_g^2 \\
&\overset{(B.14)}{=} \frac{1}{2}\int_{\partial M} \operatorname{tr}_g h \, \|\mathring{\operatorname{Ric}}\|_g^2 - \frac{1}{2}\int_{\partial M} h(\nu,\nu)\, \|\mathring{\operatorname{Ric}}\|_g^2 \\
&\quad - 2\int_{\partial M} \operatorname{Riem}^{\alpha\ \ \beta}_{\ \ \mu\ \ \nu} \mathring{\operatorname{Ric}}_{\alpha\beta} h^{\mu\nu} - \int_{\partial M} \mathring{\operatorname{Ric}}{:}\nabla^2 \operatorname{tr}_g h \\
&\quad - \int_{\partial M} \mathring{\operatorname{Ric}}{:}\Delta h + 2\int_{\partial M} \mathring{\operatorname{Ric}}{:}\nabla \operatorname{div} h. \qquad \text{(B.36)}
\end{aligned}
$$

Now, at $t = 0$, $(\partial M, \widetilde{g})\big|_{t=0}$ is an n–sphere with radius r, and we have by our choice of coordinates

$$
\begin{aligned}
\widetilde{g}_{ij}\Big|_{t=0} &= \widetilde{\eta}_{ij} \quad (= \eta_{ij} \ ), \\
\widetilde{\mathrm{Riem}}_{ijkl}\Big|_{t=0} &= \frac{1}{r^2}\left(\widetilde{\eta}_{ik}\widetilde{\eta}_{jl} - \widetilde{\eta}_{il}\widetilde{\eta}_{jk}\right), \\
\widetilde{\mathrm{Ric}}_{ij}\Big|_{t=0} &= \frac{n-1}{r^2}\widetilde{\eta}_{ij}, \\
\widetilde{\mathrm{Scal}}\Big|_{t=0} &= \frac{n(n-1)}{r^2}, \\
\widetilde{\mathring{\mathrm{Ric}}}_{ij}\Big|_{t=0} &= 0.
\end{aligned}
$$

To recover the second fundamental form, remark that ∂M is umbilical in M, i.e. $\mathring{A} = 0$, or $A = \frac{H}{n}\widetilde{\eta}$. Plugging this into the Gauss equations,

$$
\widetilde{\mathrm{Riem}}_{ijkl} = \mathrm{Riem}_{ijkl} + A_{ik}A_{jl} - A_{il}A_{jk},
$$

we conclude that

$$
\begin{aligned}
H &= n\frac{\sqrt{R^2 - r^2}}{Rr}, \\
A &= \frac{\sqrt{R^2 - r^2}}{Rr}\widetilde{\eta}.
\end{aligned}
$$

Considering the three quantities $\int_{\partial M} \|\mathring{A}\|_g^2$, $\int_{\partial M}\left(H - \overline{H}\right)^2$ and $\int_{\partial M}\|\mathring{\mathrm{Ric}}\|_g^2$, the evolutions of which are given by (B.31), (B.35) and (B.36), respectively, we see that all vanish at $t = 0$. Moreover, since these quantities are non–negative at all times, we infer from their evolutions that they must be minimal initially. We are thus interested in their second variations at $t = 0$. For this, we calculate:

$$
\begin{aligned}
\partial_t\Big|_{t=0}\mathring{A}_{ij} &\overset{\text{(B.29)}}{=} 0 + \frac{\sqrt{R^2 - r^2}}{nRr}\operatorname{tr}_{\widetilde{\eta}} h\eta_{ij} - \frac{\sqrt{R^2 - r^2}}{Rr}h_{ij} \\
&\quad - \frac{1}{2}\left(\left(\nabla h\right)(\nu)\right)_{ij} - \frac{1}{2}\left(\left(\nabla h\right)(\nu)\right)_{ji} + \frac{1}{n}\operatorname{div} h(\nu)\eta_{ij} \\
&\quad + \frac{1}{2}\left(\nabla_\nu h\right)_{ij} - \frac{1}{2n}\nabla_\nu \operatorname{tr}_\eta h\eta_{ij} - \frac{1}{2n}\left(\nabla_\nu h\right)(\nu,\nu)\eta_{ij} \\
&= -\frac{\sqrt{R^2 - r^2}}{Rr}h_{ij} + \frac{\sqrt{R^2 - r^2}}{nRr}\operatorname{tr}_\eta h\eta_{ij} - \frac{\sqrt{R^2 - r^2}}{nRr}h(\nu,\nu)\eta_{ij} \\
&\quad - \frac{1}{2}\left(\left(\nabla h\right)(\nu)\right)_{ij} - \frac{1}{2}\left(\left(\nabla h\right)(\nu)\right)_{ji} + \frac{1}{n}\operatorname{div} h(\nu)\eta_{ij} \\
&\quad + \frac{1}{2}\left(\nabla_\nu h\right)_{ij} - \frac{1}{2n}\nabla_\nu \operatorname{tr}_\eta h\eta_{ij} - \frac{1}{2n}\left(\nabla_\nu h\right)(\nu,\nu)\eta_{ij}.
\end{aligned}
\tag{B.37}
$$

Then, in view of the fact that $\mathring{A}$ vanishes at $t=0$,

$$
\begin{aligned}
\partial_t^2\Big|_{t=0}\int_{\partial M}\|\mathring{A}\|^2_{\widetilde{g}} \overset{\text{(B.31)}}{=} & -2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\widetilde{\eta}_{kl}\left(\widetilde{\eta^{-1}}\right)^{li}\left(\partial_t\Big|_{t=0}\mathring{A}_{ij}\right)\left(\widetilde{\eta^{-1}}\right)^{ks}\left(\widetilde{\eta^{-1}}\right)^{jt}h_{st} \\
& -2\int_{\partial M}\left(\partial_t\Big|_{t=0}\mathring{A}\right)\widetilde{:}\left(\left(\nabla h\right)(\nu)\right)+\int_{\partial M}\left(\partial_t\Big|_{t=0}\mathring{A}\right)\widetilde{:}\left(\nabla_\nu h\right)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\text{(B.37)}}{=} & \, 2\frac{R^2-r^2}{R^2r^2}\int_{\partial M}h\widetilde{:}h-2\frac{R^2-r^2}{nR^2r^2}\int_{\partial M}\operatorname{tr}_\eta h\operatorname{tr}_{\widetilde{\eta}}h+2\frac{R^2-r^2}{nR^2r^2}\int_{\partial M}h(\nu,\nu)\operatorname{tr}_{\widetilde{\eta}}h \\
& +2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\left(\left(\nabla h\right)(\nu)\right)\widetilde{:}h-2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\operatorname{div}h(\nu)\operatorname{tr}_{\widetilde{\eta}}h \\
& -\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\left(\nabla_\nu h\right)\widetilde{:}h+\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\left(\nabla_\nu\operatorname{tr}_\eta h\right)\operatorname{tr}_{\widetilde{\eta}}h \\
& +\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\left(\left(\nabla_\nu h\right)(\nu,\nu)\right)\operatorname{tr}_{\widetilde{\eta}}h \\
& +2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\left(\left(\nabla h\right)(\nu)\right)\widetilde{:}h-2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\left(\operatorname{tr}_{\widetilde{\eta}}\left(\nabla h\right)(\nu)\right)\operatorname{tr}_\eta h \\
& +2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\left(\operatorname{tr}_{\widetilde{\eta}}\left(\nabla h\right)(\nu)\right)h(\nu,\nu) \\
& +\int_{\partial M}\|\left(\nabla h\right)(\nu)\|^2_{\widetilde{\eta}}+\int_{\partial M}\left(\left(\nabla h\right)(\nu)\right)_{ji}\left(\widetilde{\eta^{-1}}\right)^{ik}\left(\widetilde{\eta^{-1}}\right)^{jl}\left(\left(\nabla h\right)(\nu)\right)_{kl} \\
& -\frac{2}{n}\int_{\partial M}\left(\operatorname{div}h(\nu)\right)\left(\operatorname{tr}_{\widetilde{\eta}}\left(\nabla h\right)(\nu)\right)-\int_{\partial M}\left(\nabla_\nu h\right)\widetilde{:}\left(\left(\nabla h\right)(\nu)\right) \\
& +\frac{1}{n}\int_{\partial M}\left(\nabla_\nu\operatorname{tr}_\eta h\right)\left(\operatorname{tr}_{\widetilde{\eta}}\left(\nabla h\right)(\nu)\right)+\frac{1}{n}\int_{\partial M}\left(\left(\nabla_\nu h\right)(\nu,\nu)\right)\left(\operatorname{tr}_{\widetilde{\eta}}\left(\nabla h\right)(\nu)\right) \\
& -\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\left(\nabla_\nu h\right)\widetilde{:}h+\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\left(\operatorname{tr}_{\widetilde{\eta}}\nabla_\nu h\right)\operatorname{tr}_\eta h \\
& -\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}\left(\operatorname{tr}_{\widetilde{\eta}}\nabla_\nu h\right)h(\nu,\nu) \\
& -\int_{\partial M}\left(\left(\nabla h\right)(\nu)\right)\widetilde{:}\left(\nabla_\nu h\right)+\frac{1}{n}\int_{\partial M}\left(\operatorname{div}h(\nu)\right)\left(\operatorname{tr}_{\widetilde{\eta}}\nabla_\nu h\right)+\frac{1}{2}\int_{\partial M}\|\nabla_\nu h\|^2_{\widetilde{\eta}} \\
& -\frac{1}{2n}\int_{\partial M}\left(\nabla_\nu\operatorname{tr}_\eta h\right)\left(\operatorname{tr}_{\widetilde{\eta}}\left(\nabla_\nu h\right)\right)-\frac{1}{2n}\int_{\partial M}\left(\left(\nabla_\nu h\right)(\nu,\nu)\right)\left(\operatorname{tr}_{\widetilde{\eta}}\nabla_\nu h\right).
\end{aligned}
$$

Remember that the restriction to ∂M of any two–tensor field B on M satisfies $\operatorname{tr}_{\widetilde{g}}B=\operatorname{tr}_g B-B(\nu,\nu)$. Also, it is easy to see that, for any two–tensor fields B and C on M restricted to ∂M,

$$B\widetilde{:}C = B{:}C-\operatorname{tr}_g\left(B(\nu,\cdot)C(\nu,\cdot)\right)-\operatorname{tr}_g\left(B(\cdot,\nu)C(\cdot,\nu)\right)+B(\nu,\nu)C(\nu,\nu).$$

Then, defining $X \cdot Y = g_{\mu\nu} X^\mu Y^\nu$ and $|X|_g^2 = X \cdot X$ for any vectorfields X and Y on M at any time t, we further obtain

$$
\begin{aligned}
\partial_t^2\Big|_{t=0} \int_{\partial M} \|\mathring{A}\|_{\widetilde{g}}^2 &= 2\frac{R^2-r^2}{R^2r^2}\int_{\partial M}\left(\|h\|_\eta^2 - 2\,|h(\nu)|_\eta^2 + (h(\nu,\nu))^2\right) - 2\frac{R^2-r^2}{nR^2r^2}\int_{\partial M}(\operatorname{tr}_\eta h - h(\nu,\nu))^2 \\
&\quad + 4\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\left((\nabla h)(\nu){:}h - (\nabla_\nu h)(\nu)\cdot h(\nu) - (\nabla h)(\nu,\nu)\cdot h(\nu) + (\nabla_\nu h)(\nu,\nu)h(\nu,\nu)\right) \\
&\quad - 4\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}(\operatorname{div} h(\nu))\,(\operatorname{tr}_\eta h - h(\nu,\nu)) \\
&\quad - 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\left((\nabla_\nu h){:}h - 2\,(\nabla_\nu h)(\nu)\cdot h(\nu) + (\nabla_\nu h)(\nu,\nu)h(\nu,\nu)\right) \\
&\quad + 2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}(\nabla_\nu \operatorname{tr}_\eta h)\,(\operatorname{tr}_\eta h - h(\nu,\nu)) + 2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\,(\operatorname{tr}_\eta h - h(\nu,\nu)) \\
&\quad + \int_{\partial M}\left(\|(\nabla h)(\nu)\|_\eta^2 - |(\nabla_\nu h)(\nu)|_\eta^2 - |(\nabla h)(\nu,\nu)|_\eta^2 + ((\nabla_\nu h)(\nu,\nu))^2\right) \\
&\quad + \int_{\partial M}\left(((\nabla h)(\nu))^{\beta\alpha}\,((\nabla h)(\nu))_{\alpha\beta} - 2\,(\nabla_\nu h)(\nu)\cdot(\nabla h)(\nu,\nu) + ((\nabla_\nu h)(\nu,\nu))^2\right) \\
&\quad + \frac{1}{2}\int_{\partial M}\left(\|\nabla_\nu h\|_\eta^2 - 2\,|(\nabla_\nu h)(\nu)|_\eta^2 + ((\nabla_\nu h)(\nu,\nu))^2\right) - \frac{2}{n}\int_{\partial M}(\operatorname{div} h(\nu))^2 \\
&\quad - \frac{1}{2n}\int_{\partial M}(\nabla_\nu \operatorname{tr}_\eta h)^2 + \frac{2}{n}\int_{\partial M}(\operatorname{div} h(\nu))\,((\nabla_\nu h)(\nu,\nu)) \\
&\quad + \frac{2}{n}\int_{\partial M}(\operatorname{div} h(\nu))\,(\nabla_\nu \operatorname{tr}_\eta h) - \frac{1}{n}\int_{\partial M}(\nabla_\nu \operatorname{tr}_\eta h)\,((\nabla_\nu h)(\nu,\nu)) \\
&\quad - 2\int_{\partial M}\left((\nabla h)(\nu){:}\,(\nabla_\nu h) - |(\nabla_\nu h)(\nu)|_\eta^2 - (\nabla_\nu h)(\nu)\cdot(\nabla h)(\nu,\nu) + ((\nabla_\nu h)(\nu,\nu))^2\right) \\
&\quad - \frac{1}{2n}\int_{\partial M}((\nabla_\nu h)(\nu,\nu))^2 \\
&= 2\frac{R^2-r^2}{R^2r^2}\int_{\partial M}\|h\|_\eta^2 - 4\frac{R^2-r^2}{R^2r^2}\int_{\partial M}|h(\nu)|_\eta^2 \\
&\quad + 2(n-1)\frac{R^2-r^2}{nR^2r^2}\int_{\partial M}(h(\nu,\nu))^2 + 4\frac{R^2-r^2}{nR^2r^2}\int_{\partial M}(\operatorname{tr}_\eta h)\,(h(\nu,\nu)) \\
&\quad - 2\frac{R^2-r^2}{nR^2r^2}\int_{\partial M}(\operatorname{tr}_\eta h)^2 + 4\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}(\nabla h)(\nu){:}h \\
&\quad - 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}(\nabla_\nu h){:}h - 4\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}(\nabla h)(\nu,\nu)\cdot h(\nu) \\
&\quad + 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}(\nabla_\nu h)(\nu,\nu)h(\nu,\nu) \\
&\quad - 4\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}(\operatorname{div} h(\nu))\,(\operatorname{tr}_\eta h - h(\nu,\nu)) \\
&\quad + 2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}(\nabla_\nu \operatorname{tr}_\eta h)\,(\operatorname{tr}_\eta h - h(\nu,\nu)) \\
&\quad + 2\frac{\sqrt{R^2-r^2}}{nRr}\int_{\partial M}((\nabla_\nu h)(\nu,\nu))\,(\operatorname{tr}_\eta h - h(\nu,\nu))
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\partial M} \|(\nabla h)(\nu)\|_\eta^2 + \frac{1}{2}\int_{\partial M} \|\nabla_\nu h\|_\eta^2 \\
&+ \int_{\partial M} ((\nabla h)(\nu))^{\beta\alpha} ((\nabla h)(\nu))_{\alpha\beta} - 2\int_{\partial M} ((\nabla h)(\nu)) : (\nabla_\nu h) \\
&- \int_{\partial M} |(\nabla h)(\nu,\nu)|_\eta^2 - \frac{2}{n}\int_{\partial M} (\operatorname{div} h(\nu))^2 - \frac{1}{2n}\int_{\partial M} (\nabla_\nu \operatorname{tr}_\eta h)^2 \\
&+ \frac{n-1}{2n}\int_{\partial M} ((\nabla_\nu h)(\nu,\nu))^2 + \frac{2}{n}\int_{\partial M} (\operatorname{div} h(\nu)) ((\nabla_\nu h)(\nu,\nu)) \\
&+ \frac{2}{n}\int_{\partial M} (\operatorname{div} h(\nu)) (\nabla_\nu \operatorname{tr}_\eta h) - \frac{1}{n}\int_{\partial M} (\nabla_\nu \operatorname{tr}_\eta h) ((\nabla_\nu h)(\nu,\nu)) . \qquad \text{(B.38)}
\end{aligned}
$$

Similarly, we calculate (using that $H\big|_{t=0}$ is constant on ∂M)

$$
\begin{aligned}
\partial_t\big|_{t=0} \left(H - \overline{H}\right) &\overset{\text{(B.34)}}{=} -\frac{\sqrt{R^2-r^2}}{Rr}\left(\operatorname{tr}_\eta h - \fint_{\partial M} \operatorname{tr}_\eta h\right) + \frac{\sqrt{R^2-r^2}}{Rr}\left(h(\nu,\nu) - \fint_{\partial M} h(\nu,\nu)\right) \\
&\quad + n\frac{\sqrt{R^2-r^2}}{2Rr}\left(h(\nu,\nu) - \fint_{\partial M} h(\nu,\nu)\right) - \operatorname{div} h(\nu) + \fint_{\partial M} \operatorname{div} h(\nu) \\
&\quad + \frac{1}{2}(\nabla_\nu h)(\nu,\nu) - \frac{1}{2}\fint_{\partial M} (\nabla_\nu h)(\nu,\nu) + \frac{1}{2}\nabla_\nu \operatorname{tr}_\eta h - \frac{1}{2}\fint_{\partial M} \nabla_\nu \operatorname{tr}_\eta h \\
&= -\frac{\sqrt{R^2-r^2}}{Rr}\operatorname{tr}_\eta h + \frac{\sqrt{R^2-r^2}}{Rr}\fint_{\partial M} \operatorname{tr}_\eta h \\
&\quad + (n+2)\frac{\sqrt{R^2-r^2}}{2Rr} h(\nu,\nu) - (n+2)\frac{\sqrt{R^2-r^2}}{2Rr}\fint_{\partial M} h(\nu,\nu) \\
&\quad - \operatorname{div} h(\nu) + \fint_{\partial M} \operatorname{div} h(\nu) + \frac{1}{2}(\nabla_\nu h)(\nu,\nu) \\
&\quad - \frac{1}{2}\fint_{\partial M} (\nabla_\nu h)(\nu,\nu) + \frac{1}{2}\nabla_\nu \operatorname{tr}_\eta h - \frac{1}{2}\fint_{\partial M} \nabla_\nu \operatorname{tr}_\eta h \qquad \text{(B.39)}
\end{aligned}
$$

to find

$$
\begin{aligned}
\partial_t^2\big|_{t=0} \int_{\partial M} \left(H - \overline{H}\right)^2 &\overset{\text{(B.35)}}{=} -2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} \left(\partial_t\big|_{t=0}\left(H-\overline{H}\right)\right)\operatorname{tr}_\eta h \\
&\quad + 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} \left(\partial_t\big|_{t=0}\left(H-\overline{H}\right)\right) h(\nu,\nu) \\
&\quad + n\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} \left(\partial_t\big|_{t=0}\left(H-\overline{H}\right)\right) h(\nu,\nu)
\end{aligned}
$$

$$
\begin{aligned}
&- 2 \int_{\partial M} \left(\partial_t\big|_{t=0} \left(H - \overline{H}\right)\right) (\operatorname{div} h(\nu)) \\
&+ \int_{\partial M} \left(\partial_t\big|_{t=0} \left(H - \overline{H}\right)\right) (\nabla_\nu \operatorname{tr}_\eta h) \\
&+ \int_{\partial M} \left(\partial_t\big|_{t=0} \left(H - \overline{H}\right)\right) ((\nabla_\nu h)(\nu,\nu))
\end{aligned}
$$

$$
\begin{aligned}
\overset{\text{(B.39)}}{=} & 2\frac{R^2-r^2}{R^2r^2}\int_{\partial M}(\operatorname{tr}_\eta h)^2 - 2\frac{R^2-r^2}{\widetilde{\operatorname{Vol}}R^2r^2}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)^2 - (n+2)\frac{R^2-r^2}{R^2r^2}\int_{\partial M} h(\nu,\nu)\operatorname{tr}_\eta h \\
& + (n+2)\frac{R^2-r^2}{\widetilde{\operatorname{Vol}}R^2r^2}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{tr}_\eta h\right) + 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h \operatorname{div} h(\nu) \\
& - 2\frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\operatorname{div} h(\nu)\right) - \frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\operatorname{tr}_\eta h \\
& + \frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{tr}_\eta h\right) - \frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h \nabla_\nu \operatorname{tr}_\eta h \\
& + \frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right) \\[2ex]
& - (n+2)\frac{R^2-r^2}{R^2r^2}\int_{\partial M} h(\nu,\nu)\operatorname{tr}_\eta h + (n+2)\frac{R^2-r^2}{\widetilde{\operatorname{Vol}}R^2r^2}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{tr}_\eta h\right) \\
& + (n+2)^2\frac{R^2-r^2}{2R^2r^2}\int_{\partial M}(h(\nu,\nu))^2 - (n+2)^2\frac{R^2-r^2}{2\widetilde{\operatorname{Vol}}R^2r^2}\left(\int_{\partial M} h(\nu,\nu)\right)^2 \\
& - (n+2)\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} h(\nu,\nu)\operatorname{div} h(\nu) + (n+2)\frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{div} h(\nu)\right) \\
& + (n+2)\frac{\sqrt{R^2-r^2}}{2Rr}\int_{\partial M} h(\nu,\nu)(\nabla_\nu h)(\nu,\nu) - (n+2)\frac{\sqrt{R^2-r^2}}{2\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right) \\
& + (n+2)\frac{\sqrt{R^2-r^2}}{2Rr}\int_{\partial M} h(\nu,\nu)\nabla_\nu \operatorname{tr}_\eta h - (n+2)\frac{\sqrt{R^2-r^2}}{2\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right) \\[2ex]
& + 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h \operatorname{div} h(\nu) - 2\frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\operatorname{div} h(\nu)\right) \\
& - (n+2)\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} h(\nu,\nu)\operatorname{div} h(\nu) + (n+2)\frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{div} h(\nu)\right) \\
& + 2\int_{\partial M}(\operatorname{div} h(\nu))^2 - \frac{2}{\widetilde{\operatorname{Vol}}}\left(\int_{\partial M}\operatorname{div} h(\nu)\right)^2 - \int_{\partial M}(\operatorname{div} h(\nu))((\nabla_\nu h)(\nu,\nu)) \\
& + \frac{1}{\widetilde{\operatorname{Vol}}}\left(\int_{\partial M}\operatorname{div} h(\nu)\right)\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right) - \int_{\partial M}(\operatorname{div} h(\nu))(\nabla_\nu \operatorname{tr}_\eta h) \\
& + \frac{1}{\widetilde{\operatorname{Vol}}}\left(\int_{\partial M}\operatorname{div} h(\nu)\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right) \\[2ex]
& - \frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h \nabla_\nu \operatorname{tr}_\eta h + \frac{\sqrt{R^2-r^2}}{\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right) \\
& + (n+2)\frac{\sqrt{R^2-r^2}}{2Rr}\int_{\partial M} h(\nu,\nu)\nabla_\nu \operatorname{tr}_\eta h - (n+2)\frac{\sqrt{R^2-r^2}}{2\widetilde{\operatorname{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right) \\
& - \int_{\partial M}(\operatorname{div} h(\nu))(\nabla_\nu \operatorname{tr}_\eta h) + \frac{1}{\widetilde{\operatorname{Vol}}}\left(\int_{\partial M}\operatorname{div} h(\nu)\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right) \\
& + \frac{1}{2}\int_{\partial M}(\nabla_\nu \operatorname{tr}_\eta h)((\nabla_\nu h)(\nu,\nu)) - \frac{1}{2\widetilde{\operatorname{Vol}}}\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right)\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)
\end{aligned}
$$

$$+\frac{1}{2}\int_{\partial M}(\nabla_\nu \operatorname{tr}_\eta h)^2-\frac{1}{2\widetilde{\mathrm{Vol}}}\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right)^2$$

$$-\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h\,(\nabla_\nu h)(\nu,\nu)+\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)$$
$$+(n+2)\frac{\sqrt{R^2-r^2}}{2Rr}\int_{\partial M}h(\nu,\nu)\,(\nabla_\nu h)(\nu,\nu)-(n+2)\frac{\sqrt{R^2-r^2}}{2\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M}h(\nu,\nu)\right)\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)$$
$$-\int_{\partial M}(\operatorname{div}h(\nu))\,((\nabla_\nu h)(\nu,\nu))+\frac{1}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M}\operatorname{div}h(\nu)\right)\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)$$
$$+\frac{1}{2}\int_{\partial M}((\nabla_\nu h)(\nu,\nu))^2-\frac{1}{2\widetilde{\mathrm{Vol}}}\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)^2$$
$$+\frac{1}{2}\int_{\partial M}((\nabla_\nu h)(\nu,\nu))\,(\nabla_\nu \operatorname{tr}_\eta h)-\frac{1}{2\widetilde{\mathrm{Vol}}}\left(\int_{\partial M}(\nabla_\nu h)(\nu,\nu)\right)\left(\int_{\partial M}\nabla_\nu \operatorname{tr}_\eta h\right)$$

$$=2\frac{R^2-r^2}{R^2r^2}\int_{\partial M}(\operatorname{tr}_\eta h)^2-2\frac{R^2-r^2}{\widetilde{\mathrm{Vol}}R^2r^2}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)^2$$
$$-2(n+2)\frac{R^2-r^2}{R^2r^2}\int_{\partial M}h(\nu,\nu)\operatorname{tr}_\eta h$$
$$+2(n+2)\frac{R^2-r^2}{\widetilde{\mathrm{Vol}}R^2r^2}\left(\int_{\partial M}h(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{tr}_\eta h\right)$$
$$+(n+2)^2\frac{R^2-r^2}{2R^2r^2}\int_{\partial M}(h(\nu,\nu))^2$$
$$-(n+2)^2\frac{R^2-r^2}{2\widetilde{\mathrm{Vol}}R^2r^2}\left(\int_{\partial M}h(\nu,\nu)\right)^2$$

$$
\begin{aligned}
&+ 4\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h\operatorname{div} h(\nu)\\
&- 4\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\operatorname{div} h(\nu)\right)\\
&- 2(n+2)\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} h(\nu,\nu)\operatorname{div} h(\nu)\\
&+ 2(n+2)\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\operatorname{div} h(\nu)\right)\\
&- 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h\nabla_\nu\operatorname{tr}_\eta h\\
&+ 2\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\nabla_\nu\operatorname{tr}_\eta h\right)\\
&+ (n+2)\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} h(\nu,\nu)\nabla_\nu\operatorname{tr}_\eta h\\
&- (n+2)\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\nabla_\nu\operatorname{tr}_\eta h\right)\\
&- 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M}\operatorname{tr}_\eta h\left(\nabla_\nu h\right)(\nu,\nu)\\
&+ 2\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M}\operatorname{tr}_\eta h\right)\left(\int_{\partial M}\left(\nabla_\nu h\right)(\nu,\nu)\right)\\
&+ (n+2)\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} h(\nu,\nu)\left(\nabla_\nu h\right)(\nu,\nu)\\
&- (n+2)\frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}}Rr}\left(\int_{\partial M} h(\nu,\nu)\right)\left(\int_{\partial M}\left(\nabla_\nu h\right)(\nu,\nu)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2 \int_{\partial M} (\operatorname{div} h(\nu))^2 - \frac{2}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \operatorname{div} h(\nu) \right)^2 \\
&+ \frac{1}{2} \int_{\partial M} (\nabla_\nu \operatorname{tr}_\eta h)^2 - \frac{1}{2\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu \operatorname{tr}_\eta h \right)^2 \\
&+ \frac{1}{2} \int_{\partial M} ((\nabla_\nu h)(\nu,\nu))^2 - \frac{1}{2\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} (\nabla_\nu h)(\nu,\nu) \right)^2 \\
&- 2 \int_{\partial M} (\operatorname{div} h(\nu)) (\nabla_\nu \operatorname{tr}_\eta h) \\
&+ \frac{2}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \operatorname{div} h(\nu) \right) \left(\int_{\partial M} \nabla_\nu \operatorname{tr}_\eta h \right) \\
&- 2 \int_{\partial M} (\operatorname{div} h(\nu)) ((\nabla_\nu h)(\nu,\nu)) \\
&+ \frac{2}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \operatorname{div} h(\nu) \right) \left(\int_{\partial M} (\nabla_\nu h)(\nu,\nu) \right) \\
&+ \int_{\partial M} ((\nabla_\nu h)(\nu,\nu)) (\nabla_\nu \operatorname{tr}_\eta h) \\
&- \frac{1}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} (\nabla_\nu h)(\nu,\nu) \right) \left(\int_{\partial M} \nabla_\nu \operatorname{tr}_\eta h \right). \qquad \text{(B.40)}
\end{aligned}
$$

Finally, and completely analogously to the derivation of (B.19), we obtain

$$
\begin{aligned}
\partial_t^2\Big|_{t=0} \int_{\partial M} \|\mathring{\mathrm{Ric}}\|_g^2 \overset{(\mathrm{B.36})}{=} & -2 \int_{\partial M} \left(\mathrm{Riem}^{\alpha\ \ \beta}_{\ \ \mu\ \ \nu}\right)\Big|_{t=0} \left(\partial_t\big|_{t=0} \mathring{\mathrm{Ric}}_{\alpha\beta}\right) h^{\mu\nu} \\
& - \int_{\partial M} \left(\partial_t\big|_{t=0} \mathring{\mathrm{Ric}}_{\alpha\beta}\right) \nabla^\alpha \nabla^\beta \operatorname{tr}_\eta h - \int_{\partial M} \left(\partial_t\big|_{t=0} \mathring{\mathrm{Ric}}_{\alpha\beta}\right) \Delta h^{\alpha\beta} \\
& + 2 \int_{\partial M} \left(\partial_t\big|_{t=0} \mathring{\mathrm{Ric}}_{\alpha\beta}\right) \nabla^\alpha \operatorname{div} h^\beta \\
\overset{(\mathrm{B.18})}{=} & \frac{2}{R^4} \int_{\partial M} \mathring{h}{:}h - \frac{2}{R^2} \int_{\partial M} \mathring{h}{:}\nabla^2 \operatorname{tr}_\eta h \\
& - \frac{2}{R^2} \int_{\partial M} \mathring{h}{:}\Delta h + \frac{4}{R^2} \int_{\partial M} \mathring{h}{:}\nabla \operatorname{div} h \\
& + \frac{1}{2} \int_{\partial M} \left\|\nabla^2 \operatorname{tr}_\eta h\right\|_\eta^2 + \frac{1}{2} \int_{\partial M} \|\Delta h\|_\eta^2 + \int_{\partial M} \|\nabla \operatorname{div} h\|_\eta^2 \\
& - \frac{2}{n+1} \int_{\partial M} (\Delta \operatorname{tr}_\eta h)^2 - \frac{2}{n+1} \int_{\partial M} (\operatorname{div}\operatorname{div} h)^2 \\
& + \int_{\partial M} \nabla^2 \operatorname{tr}_\eta h{:}\Delta h - 2 \int_{\partial M} \nabla^2 \operatorname{tr}_\eta h{:}\nabla \operatorname{div} h \\
& - 2 \int_{\partial M} \Delta h{:}\nabla \operatorname{div} h + \int_{\partial M} \nabla^\beta \operatorname{div} h^\alpha \nabla_\alpha \operatorname{div} h_\beta \\
& + \frac{4}{n+1} \int_{\partial M} (\Delta \operatorname{tr}_g h)(\operatorname{div}\operatorname{div} h) .
\end{aligned} \tag{B.41}
$$

4. The special case $h = fg$

For f a smooth function, the ansatz $h = fg$ implicates on M:

$$
\begin{aligned}
&\operatorname{tr}_g h = (n+1)f, \qquad \mathring{h} = 0, \qquad \nabla^2 \operatorname{tr}_g h = (n+1)\operatorname{Hess} f, \qquad \Delta \operatorname{tr}_g h = (n+1)\Delta f, \\
&\Delta h = (\Delta f)\, g, \qquad \operatorname{div} h = \nabla f, \qquad \nabla \operatorname{div} h = \operatorname{Hess} f, \qquad \operatorname{div}\operatorname{div} h = \Delta f,
\end{aligned}
$$

so that

$$
\begin{aligned}
\left\|\nabla^2 \operatorname{tr}_g h\right\|_g^2 &= (n+1)^2 \|\operatorname{Hess} f\|_g^2 , & \|\Delta h\|_g^2 &= (n+1)(\Delta f)^2 , \\
\nabla^2 \operatorname{tr}_g h{:}\Delta h &= (n+1)(\Delta f)^2 , & \nabla^2 \operatorname{tr}_g h{:}\nabla \operatorname{div} h &= (n+1)\|\operatorname{Hess} f\|_g^2 , \\
\Delta h{:}\nabla \operatorname{div} h &= (\Delta f)^2 , & \nabla^\beta \operatorname{div} h^\alpha \nabla_\alpha \operatorname{div} h_\beta &= \|\operatorname{Hess} f\|_g^2 .
\end{aligned}
$$

It follows that, in this case,

$$
\begin{aligned}
\partial_t^2\Big|_{t=0} \int_M \|\mathring{\mathrm{Ric}}\|_g^2 &\overset{(B.19)}{=} 0 - 0 - 0 + 0 \\
&\quad + \frac{1}{2}(n+1)^2 \int_M \|\mathrm{Hess}\, f\|_\eta^2 + \frac{1}{2}(n+1) \int_M (\Delta f)^2 \\
&\quad + \int_M \|\mathrm{Hess}\, f\|_\eta^2 - \frac{2}{n+1}(n+1)^2 \int_M (\Delta f)^2 \\
&\quad - \frac{2}{n+1} \int_M (\Delta f)^2 + (n+1) \int_M (\Delta f)^2 \\
&\quad - 2(n+1) \int_M \|\mathrm{Hess}\, f\|_\eta^2 - 2 \int_M (\Delta f)^2 \\
&\quad + \int_M \|\mathrm{Hess}\, f\|_\eta^2 + \frac{4}{n+1}(n+1) \int_M (\Delta f)^2 \\
&= \frac{1}{2}(n-1)^2 \left(\int_M \|\mathrm{Hess}\, f\|_\eta^2 - \frac{1}{n+1} \int_M (\Delta f)^2 \right) \\
&= \frac{1}{2}(n-1)^2 \int_M \|\mathring{\mathrm{Hess}} f\|_\eta^2 , \qquad \text{(B.42)}
\end{aligned}
$$

where $\mathring{\mathrm{Hess}} f$ denotes the traceless part of $\mathrm{Hess}\, f$. Similarly,

$$
\begin{aligned}
\partial_t^2\Big|_{t=0} \int_M \left(\mathrm{Scal} - \overline{\mathrm{Scal}}\right)^2 &\overset{(B.21)}{=} \frac{2n^2}{R^4}(n+1)^2 \int_M f^2 + \frac{4n}{R^2}(n+1)^2 \int_M f\Delta f \\
&\quad - \frac{4n}{R^2}(n+1) \int_M f\Delta f + 2(n+1)^2 \int_M (\Delta f)^2 \\
&\quad - \frac{2}{\mathrm{Vol}}(n+1)^2 \left(\int_M \Delta f \right)^2 + 2 \int_M (\Delta f)^2 \\
&\quad - \frac{2}{\mathrm{Vol}} \left(\int_M \Delta f \right)^2 - 4(n+1) \int_M (\Delta f)^2 \\
&\quad + \frac{4}{\mathrm{Vol}}(n+1) \left(\int_M \Delta f \right)^2 \\
&= \frac{2n^2}{R^4}(n+1)^2 \int_M f^2 + \frac{4n^2}{R^2}(n+1) \int_M f\Delta f \\
&\quad + 2n^2 \int_M (\Delta f)^2 - \frac{2n^2}{\mathrm{Vol}} \left(\int \Delta f \right)^2 \\
&\overset{(B.1)}{=} 2n^2 \int_M \left(\left(\frac{n+1}{R} f + \Delta f \right) - \overline{\left(\frac{n+1}{R} f + \Delta f \right)} \right)^2 . \qquad \text{(B.43)}
\end{aligned}
$$

On the other hand, the ansatz $h = fg$ yields on the boundary ∂M:

$$\operatorname{tr}_g h = (n+1)f, \qquad \nabla_\nu \operatorname{tr}_g h = (n+1)\nabla_\nu f,$$
$$h(\nu) = f\nu^\flat, \qquad h(\nu,\nu) = f,$$
$$\nabla h = (\nabla f)\, g, \qquad (\nabla h)(\nu) = (\nabla f)\,\nu^\flat, \qquad (\nabla h)(\nu,\nu) = \nabla f,$$
$$\nabla_\nu h = (\nabla_\nu f)\, g, \qquad (\nabla_\nu h)(\nu,\nu) = \nabla_\nu f, \qquad \operatorname{div} h(\nu) = \nabla_\nu f,$$

where $\nu^\flat$ is defined as $(\nu^\flat)_\alpha = g_{\alpha\beta}\nu^\beta$. Consequently,

$$\|h\|_g^2 = (n+1)f^2, \qquad |h(\nu)|_g^2 = f^2,$$
$$\|(\nabla h)(\nu)\|_g^2 = |\nabla f|_g^2, \qquad (\nabla h(\nu))^{\beta\alpha}(\nabla h(\nu))_{\alpha\beta} = (\nabla_\nu f)^2,$$
$$(\nabla h)(\nu){:}h = f\nabla_\nu f, \qquad (\nabla h)(\nu){:}\nabla_\nu h = (\nabla_\nu f)^2,$$
$$(\nabla h)(\nu,\nu)\cdot h(\nu) = f\nabla_\nu f, \qquad |(\nabla h)(\nu,\nu)|_g^2 = |\nabla f|_g^2,$$
$$\|\nabla_\nu h\|_g^2 = (n+1)(\nabla_\nu f)^2, \qquad (\nabla_\nu h){:}h = (n+1)f\nabla_\nu f.$$

It then follows that,

$$\begin{aligned}
\partial_t^2\Big|_{t=0}\int_{\partial M}\|\mathring{A}\|_{\widetilde{g}}^2 \overset{\text{(B.38)}}{=} {} & 2(n+1)\frac{R^2-r^2}{R^2r^2}\int_{\partial M} f^2 - 4\frac{R^2-r^2}{R^2r^2}\int_{\partial M} f^2 \\
& + 2\frac{n-1}{n}\frac{R^2-r^2}{R^2r^2}\int_{\partial M} f^2 + 4\frac{n+1}{n}\frac{R^2-r^2}{R^2r^2}\int_{\partial M} f^2 \\
& - 2\frac{(n+1)^2}{n}\frac{R^2-r^2}{R^2r^2}\int_{\partial M} f^2 \\
& + 4\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f - 2(n+1)\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f \\
& - 4\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f + 2\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f \\
& - 4\frac{n}{n}\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f + 2\frac{n(n+1)}{n}\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f \\
& + 2\frac{n}{n}\frac{\sqrt{R^2-r^2}}{Rr}\int_{\partial M} f\nabla_\nu f \\
& + \int_{\partial M}|\nabla f|_{\widetilde{\eta}}^2 + \frac{1}{2}(n+1)\int_{\partial M}(\nabla_\nu f)^2 + \int_{\partial M}(\nabla_\nu f)^2 \\
& - 2\int_{\partial M}(\nabla_\nu f)^2 - \int_{\partial M}|\nabla f|_{\widetilde{\eta}}^2 - \frac{2}{n}\int_{\partial M}(\nabla_\nu f)^2 \\
& - \frac{1}{2n}(n+1)^2\int_{\partial M}(\nabla_\nu f)^2 + \frac{n-1}{2n}\int_{\partial M}(\nabla_\nu f)^2 + \frac{2}{n}\int_{\partial M}(\nabla_\nu f)^2 \\
& + \frac{2}{n}(n+1)\int_{\partial M}(\nabla_\nu f)^2 - \frac{1}{n}(n+1)\int_{\partial M}(\nabla_\nu f)^2
\end{aligned}$$

$$= 0. \tag{B.44}$$

Similarly,

$$
\begin{aligned}
\partial_t^2\Big|_{t=0} \int_{\partial M} \left(H - \overline{H}\right)^2 \overset{\text{(B.40)}}{=} {} & 2(n+1)^2 \frac{R^2-r^2}{R^2r^2} \int_{\partial M} f^2 - 2(n+1)^2 \frac{R^2-r^2}{\widetilde{\mathrm{Vol}} R^2 r^2} \left(\int_{\partial M} f\right)^2 \\
& - 2(n+1)(n+2) \frac{R^2-r^2}{R^2r^2} \int_{\partial M} f^2 + 2(n+1)(n+2) \frac{R^2-r^2}{\widetilde{\mathrm{Vol}} R^2 r^2} \left(\int_{\partial M} f\right)^2 \\
& + \frac{(n+2)^2}{2} \frac{R^2-r^2}{R^2r^2} \int_{\partial M} f^2 - \frac{(n+2)^2}{2} \frac{R^2-r^2}{\widetilde{\mathrm{Vol}} R^2 r^2} \left(\int_{\partial M} f\right)^2 \\
& + 4(n+1) \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f - 4(n+1) \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& - 2(n+2) \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f + 2(n+2) \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& - 2(n+1)^2 \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f + 2(n+1)^2 \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& + (n+1)(n+2) \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f \\
& - (n+1)(n+2) \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& - 2(n+1) \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f + 2(n+1) \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& + (n+2) \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f - (n+2) \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& + 2 \int_{\partial M} (\nabla_\nu f)^2 - \frac{2}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2 + \frac{(n+1)^2}{2} \int_{\partial M} (\nabla_\nu f)^2 \\
& - \frac{(n+1)^2}{2\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2 + \frac{1}{2} \int_{\partial M} (\nabla_\nu f)^2 - \frac{1}{2\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2 \\
& - 2(n+1) \int_{\partial M} (\nabla_\nu f)^2 + 2\frac{n+1}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2 - 2 \int_{\partial M} (\nabla_\nu f)^2 \\
& + \frac{2}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2 + (n+1) \int_{\partial M} (\nabla_\nu f)^2 - \frac{n+1}{\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2 \\
= {} & \frac{n^2}{2} \frac{R^2-r^2}{R^2r^2} \int_{\partial M} f^2 - \frac{n^2}{2} \frac{R^2-r^2}{\widetilde{\mathrm{Vol}} R^2 r^2} \left(\int_{\partial M} f\right)^2 \\
& - n^2 \frac{\sqrt{R^2-r^2}}{Rr} \int_{\partial M} f \nabla_\nu f \\
& + n^2 \frac{\sqrt{R^2-r^2}}{\widetilde{\mathrm{Vol}} Rr} \left(\int_{\partial M} f\right) \left(\int_{\partial M} \nabla_\nu f\right) \\
& + \frac{n^2}{2} \int_{\partial M} (\nabla_\nu f)^2 - \frac{n^2}{2\widetilde{\mathrm{Vol}}} \left(\int_{\partial M} \nabla_\nu f\right)^2
\end{aligned}
$$

(B.45)
$$
= \frac{n^2}{2} \int_{\partial M} \left(\left(\frac{\sqrt{R^2-r^2}}{Rr} f - \nabla_\nu f\right) - \overline{\left(\frac{\sqrt{R^2-r^2}}{Rr} f - \nabla_\nu f\right)} \right)^2 .
$$

Finally, in exactly the same way we obtained (B.42), we have

$$\begin{aligned}\partial_t^2\Big|_{t=0}\int_{\partial M}\|\mathring{\mathrm{Ric}}\|_g^2 &\overset{(B.41)}{=} \frac{1}{2}(n-1)^2\left(\int_{\partial M}\|\mathrm{Hess}\, f\|_\eta^2-\frac{1}{n+1}\int_{\partial M}(\Delta f)^2\right)\\ &= \frac{1}{2}(n-1)^2\int_{\partial M}\|\mathring{\mathrm{Hess}} f\|_\eta^2\,. \end{aligned} \tag{B.46}$$

Bibliography

[And94] Ben Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171. MR 1385524 (97b:53012) (cited on page xiv)

[Aub98] Thierry Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR MR1636569 (99i:58001) (cited on pages xxi and 46)

[Bar95] Robert G. Bartle, *The elements of integration and Lebesgue measure*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1995, Containing a corrected reprint of the 1966 original [*The elements of integration*, Wiley, New York; MR0200398 (34 #293)], A Wiley-Interscience Publication. MR 1312157 (95k:28001) (cited on page xxi)

[Ber03] Marcel Berger, *A panoramic view of Riemannian geometry*, Springer-Verlag, Berlin, 2003. MR 2002701 (2004h:53001) (cited on page xxi)

[Bes87] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR 867684 (88f:53087) (cited on page xxi)

[BF36] Herbert Busemann and Willy Feller, *Krümmungseigenschaften Konvexer Flächen*, Acta Math. **66** (1936), no. 1, 1–47. MR 1555408 (cited on page xiv)

[BG92] Marcel Berger and Bernard Gostiaux, *Géométrie différentielle: variétés, courbes et surfaces*, second ed., Mathématiques. [Mathematics], Presses Universitaires de France, Paris, 1992. MR 1207362 (93j:53001) (cited on page xxi)

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038) (cited on page 32)

[Bor67] Yuriĭ E. Borovskiĭ, *Convex surfaces with a quasiconformal spherical mapping*, Sibirsk. Mat. Ž. **8** (1967), 535–547. MR 0214759 (35 #5608) (cited on page xiv)

[Bor68] ———, *An estimate for convex surfaces with a quasiconformal spherical mapping*, Sibirsk. Mat. Ž. **9** (1968), 530–535. MR 0227399 (37 #2983) (cited on page xiv)

[Bou90] Joseph V. Boussinesq, *Cours d'analyse infinitésimale: à l'usage des personnes qui étudient cette science en vue de ses applications mécaniques et physiques*, vol. I-II, Gauthier-Villard et fils, Paris, 1887-1890. (cited on page xvi)

[Bre83] Haïm Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983, Théorie et applications. [Theory and applications]. MR 697382 (85a:46001) (cited on pages xxi, 40 and 77)

[Car65] Constantin Carathéodory, *Calculus of variations and partial differential equations of the first order. Part I: Partial differential equations of the first order*, Translated by Robert B. Dean and Julius J. Brandstatter, Holden-Day Inc., San Francisco, 1965. MR MR0192372 (33 #597) (cited on page 60)

[Cha06] Isaac Chavel, *Riemannian geometry*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006, A modern introduction. MR 2229062 (2006m:53002) (cited on pages 32, 33 and 42)

[CL57] Shiing-shen Chern and Richard K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318. MR 0084811 (18,927a) (cited on page 22)

[CLN06] Bennett Chow, Peng Lu, and Lei Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 2006. MR 2274812 (2008a:53068) (cited on pages xix and 46)

[CZ56] Alberto P. Calderón and Antoni Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309. MR 0084633 (18,894a) (cited on page xvii)

[Dar96] Gaston Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, vol. 1-4, Gauthier-Villard, Paris, 1887-1896. (cited on page xvi)

[Dar12] ———, *Notice historique sur le général Meusnier*, Éloges académiques et discours: vol. publ. par le Comité du jubilé scientifique de M. Gaston Darboux, A. Hermann et fils, Paris, 1912, pp. 218–262. (cited on page xvi)

[dC76] Manfredo P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976, Translated from the Portuguese. MR 0394451 (52 #15253) (cited on pages xiii and xxi)

[dC92] ———, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty. MR MR1138207 (92i:53001) (cited on page xxi)

[Des92] Sharief Deshmukh, *Hypersurfaces of nonnegative Ricci curvature in a Euclidean space*, J. Geom. **45** (1992), no. 1-2, 48–50. MR 1188097 (93m:53056) (cited on page 48)

[DFN85] Boris A. Dubrovin, Anatoliĭ T. Fomenko, and Sergeĭ P. Novikov, *Modern geometry—methods and applications. Part II*, Graduate Texts in Mathematics, vol. 104, Springer-Verlag, New York, 1985, The geometry and topology of manifolds, Translated from the Russian by Robert G. Burns. MR 807945 (86m:53001) (cited on page 49)

[DFN90] ———, *Modern geometry—methods and applications. Part III*, Graduate Texts in Mathematics, vol. 124, Springer-Verlag, New York, 1990, Introduction to homology theory, Translated from the Russian by Robert G. Burns. MR 1076994 (91j:55001) (cited on page 50)

[Dis71] V. I. Diskant, *Certain estimates for convex surfaces with a bounded curvature function*, Sibirsk. Mat. Ž. **12** (1971), 109–125. MR 0284954 (44 #2178) (cited on page xiv)

[DLM05] Camillo De Lellis and Stefan Müller, *Optimal rigidity estimates for nearly umbilical surfaces*, J. Differential Geom. **69** (2005), no. 1, 75–110. MR MR2169583 (2006e:53078) (cited on pages viii, ix, xv, xvi, 50, 65 and 66)

[DLM06] ———, *A C^0 estimate for nearly umbilical surfaces*, Calc. Var. Partial Differential Equations **26** (2006), no. 3, 283–296. MR 2232206 (2007d:53003) (cited on page xv)

[DLT10] Camillo De Lellis and Peter M. Topping, *Almost–Schur lemma*, ArXiv e-prints (2010), arXiv:1003.3527v1 [math.DG]. (cited on pages xvii, xviii, xix, 45, 46 and 83)

[EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR 1158660 (93f:28001) (cited on page xxi)

[Eis20] Luther P. Eisenhart, *Darboux's Anteil an der Geometrie*, Acta Math. **42** (1920), no. 1, 275–284. MR 1555167 (cited on page xvi)

[Eul67] Leonhard Euler, *Recherches sur la courbure des surfaces*, Histoire de l'académie royale des sciences et belles lettres de Berlin **16** (1767), 119–143. (cited on page xvi)

[Eva98] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR MR1625845 (99e:35001) (cited on pages xxi, 9, 15, 28, 77 and 79)

[Fed69] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR 0257325 (41 #1976) (cited on pages 33, 42 and 49)

[Fet63] Abram I. Fet, *Stability theorems for convex, almost spherical surfaces*, Dokl. Akad. Nauk SSSR **153** (1963), 537–539. MR 0157292 (28 #527) (cited on page xiv)

[Ger90] Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299–314. MR 1064876 (91k:53016) (cited on page 51)

[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, third ed., Universitext, Springer-Verlag, Berlin, 2004. MR 2088027 (2005e:53001) (cited on page xxi)

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR MR1814364 (2001k:35004) (cited on pages xxi, 9 and 28)

[Gug69] Heinrich W. Guggenheimer, *Nearly spherical surfaces*, Aequationes Math. **3** (1969), 186–193. MR 0247595 (40 #859) (cited on page xiv)

[Har47] Philip Hartman, *Systems of total differential equations and Liouville's theorem on conformal mappings*, Amer. J. Math. **69** (1947), 327–332. MR 0021395 (9,59h) (cited on page xiv)

[HI08] Gerhard Huisken and Tom Ilmanen, *Higher regularity of the inverse mean curvature flow*, J. Differential Geom. **80** (2008), no. 3, 433–451. MR 2472479 (2010c:53097) (cited on page 50)

[Hil20] David Hilbert, *Gaston Darboux*, Acta Math. **42** (1920), no. 1, 269–273, 1842–1917. MR 1555166 (cited on page xvi)

[Jos08] Jürgen Jost, *Riemannian geometry and geometric analysis*, fifth ed., Universitext, Springer-Verlag, Berlin, 2008. MR 2431897 (2009g:53036) (cited on page xxi)

[Kou71] Dimitri Koutroufiotis, *Ovaloids which are almost spheres*, Comm. Pure Appl. Math. **24** (1971), 289–300. MR 0282318 (43 #8030) (cited on page xiv)

[Küh08] Wolfgang Kühnel, *Differentialgeometrie*, fourth ed., Vieweg Studium: Aufbaukurs Mathematik. [Vieweg Studies: Mathematics Course], Vieweg, Wiesbaden, 2008, Kurven–Flächen–Mannigfaltigkeiten. [Curves–surfaces–manifolds]. MR 2527182 (2010k:53001) (cited on page xiii)

[Lee97] John M. Lee, *Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer-Verlag, New York, 1997, An introduction to curvature. MR 1468735 (98d:53001) (cited on page xxi)

[Lei99] Kurt Leichtweiß, *Nearly umbilical ovaloids in the n-space are close to spheres*, Results Math. **36** (1999), no. 1-2, 102–109. MR 1706540 (2000h:52007) (cited on page xiv)

[LL97] Elliott H. Lieb and Michael Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997. MR 1415616 (98b:00004) (cited on pages 9 and 28)

[LM10] Tobias Lamm and Jan Metzger, *Small surfaces of Willmore type in Riemannian manifolds*, Int. Math. Res. Not. IMRN (2010), no. 19, 3786–3813. MR 2725514 (cited on page xv)

[LMS09] Tobias Lamm, Jan Metzger, and Felix Schulze, *Foliations of asymptotically flat manifolds by surfaces of Willmore type*, ArXiv e-prints (2009), arXiv:0903.1277v1 [math.DG]. (cited on page xv)

[MdL85] Jean-Baptiste-Marie-Charles Meusnier de Laplace, *Mémoire sur la courbure des surfaces*, Mémoires de mathématique et de physique, presentés à l'Académie royale des sciences, par divers sçavans & lûs dans ses assemblées, vol. 10, Académie royale des sciences (France), Paris, 1785, pp. 477–513. (cited on page xvi)

[Met07] Jan Metzger, *Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature*, J. Differential Geom. **77** (2007), no. 2, 201–236. MR 2355784 (2008j:53042) (cited on page xv)

[Mil97] John W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by David W. Weaver, Revised reprint of the 1965 original. MR 1487640 (98h:57051) (cited on page xxi)

[Mon00] Gaspard Monge, *Feuilles d'analyse appliquée à la géométrie: à l'usage de l'école polytechnique*, Thermidor, Paris, 1800, first published in 1795. (cited on page xvi)

[Mon50] ———, *Application de l'analyse à la géométrie*, 5 ed., Bachelier, Paris, 1850. (cited on page xvi)

[Moo73] John D. Moore, *Almost spherical convex hypersurfaces*, Trans. Amer. Math. Soc. **180** (1973), 347–358. MR 0320964 (47 #9497) (cited on page xiv)

[Nev69] N. S. Nevmeržickiĭ, *Stability in the umbilical surface theorem. I*, Vestnik Leningrad. Univ. **24** (1969), no. 7, 55–60. MR 0251672 (40 #4899) (cited on page xiv)

[Nic07a] Liviu I. Nicolaescu, *An invitation to Morse theory*, Universitext, Springer, New York, 2007. MR 2298610 (2009m:58023) (cited on page 50)

[Nic07b] ———, *Lectures on the geometry of manifolds*, second ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007, see also `http://www.nd.edu/~lnicolae/`. MR 2363924 (2008g:53001) (cited on page xxi)

[O'N83] Barrett O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, With applications to relativity. MR MR719023 (85f:53002) (cited on page xxi)

[Pau08] Arno Pauly, *Flächen mit lauter Nabelpunkten*, Elem. Math. **63** (2008), no. 3, 141–144. MR 2424898 (2009d:53007) (cited on page xiv)

[Pog67] Aleksei V. Pogorelov, *Nearly spherical surfaces*, J. Analyse Math. **19** (1967), 313–321. MR 0215263 (35 #6105) (cited on page xiv)

[Pog73] ———, *Extrinsic geometry of convex surfaces*, American Mathematical Society, Providence, R.I., 1973, Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 35. MR MR0346714 (49 #11439) (cited on pages xiv, 19 and 21)

[Pre10] Andrew Pressley, *Elementary differential geometry*, second ed., Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2010. MR 2598317 (2011b:53003) (cited on page xiii)

[Res68] Yuriĭ G. Reshetnyak, *Certain estimates for almost umbilical surfaces*, Sibirsk. Mat. Ž. **9** (1968), 903–917. MR 0235496 (38 #3805) (cited on page xiv)

[Res94] ———, *Stability theorems in geometry and analysis*, Mathematics and its Applications, vol. 304, Kluwer Academic Publishers Group, Dordrecht, 1994, Translated from the 1982 Russian original by N. S. Dairbekov and V. N. Dyatlov, and revised by the author, Translation edited and with a foreword by S. S. Kutateladze. MR MR1326375 (96i:30016) (cited on pages xiv, xv, xvi and xvii)

[Roc70] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR 0274683 (43 #445) (cited on pages xxi and 42)

[Rud87] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157 (88k:00002) (cited on pages xxi and 9)

[Rud91] ———, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR 1157815 (92k:46001) (cited on pages xxi, 8 and 28)

[RV74] A. Wayne Roberts and Dale E. Varberg, *Another proof that convex functions are locally Lipschitz*, Amer. Math. Monthly **81** (1974), 1014–1016. MR 0352371 (50 #4858) (cited on pages 23, 25 and 27)

[Sch88] Rolf Schneider, *Closed convex hypersurfaces with curvature restrictions*, Proc. Amer. Math. Soc. **103** (1988), no. 4, 1201–1204. MR 955009 (90a:53010) (cited on page xiv)

[Sch89] ———, *Stability in the Aleksandrov-Fenchel-Jessen theorem*, Mathematika **36** (1989), no. 1, 50–59. MR 1014200 (90h:52015) (cited on page xiv)

[Sch93] ———, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993. MR 1216521 (94d:52007) (cited on pages xxi, 22, 24, 27 and 39)

[Spi65] Michael Spivak, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965. MR 0209411 (35 #309) (cited on page xxi)

[Spi99] ———, *A comprehensive introduction to differential geometry. Vol. IV*, third ed., Publish or Perish Inc., Huston, Texas, 1999. (cited on pages xiii and xiv)

[Ste70] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR MR0290095 (44 #7280) (cited on page 14)

[Str33] Dirk J. Struik, *Outline of a history of differential geometry: I*, Isis **19** (1933), no. 1, pp. 92–120 (English). (cited on page xvi)

[Str88] ———, *Lectures on classical differential geometry*, second ed., Dover Publications Inc., New York, 1988. MR 939369 (89b:53002) (cited on page xiii)

[Tru96] Clifford Truesdell, *Jean-Baptiste-Marie Charles Meusnier de la Place* [sic!] *(1754–1793): an historical note*, Meccanica **31** (1996), no. 5, 607–610. MR 1420154 (cited on page xvi)

[Urb90] John I. E. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 3, 355–372. MR 1082861 (92c:53037) (cited on page 51)

[Urb91] ———, *An expansion of convex hypersurfaces*, J. Differential Geom. **33** (1991), no. 1, 91–125. MR 1085136 (91j:58155) (cited on page 53)

[Vod70] Sergej K. Vodop′yanov, *Estimates of the deviation of quasi-umbilical surfaces from a sphere*, Sibirsk. Mat. Ž. **11** (1970), 971–987, 1195. MR 0298603 (45 #7655) (cited on page xiv)

[Vol63] Yuriĭ A. Volkov, *Stability of the solution of Minkowski's problem*, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom **18** (1963), no. 1, 33–43. MR 0146738 (26 #4258) (cited on page xiv)

Curriculum Vitae

Geboren am 2. Mai 1978 in Wien (Österreich). Besuch des Lycée Français de Vienne von 1984 bis 1996. Juni 1996: Baccalauréat Général (Série “S”). Studium der Physik an der ETH Zürich von 1997 bis 2003. April 2003: Diplom als Physiker, Diplomarbeit *On the Newtonian Limit of General Relativity*, betreut durch Prof. Dr. Demetrios Christodoulou. Assistent am Mathematikdepartement der ETH Zürich von 2003 bis 2006. Seit März 2006: Assistent am Institut für Mathematik der Universität Zürich und Doktoratsstudium unter der Leitung von Prof. Dr. Camillo De Lellis.

Danksagung

Zu tiefstem Dank bin ich meinem Doktorvater, Camillo De Lellis, verpflichtet. Nicht nur dafür, dass er mir überhaupt die Chance gab, mich als Physiker in der reinen Mathematik zu versuchen, sondern auch für die zahlreichen Hilfestellungen, sowie seine so lang anhaltende Geduld — es ist mir bewusst, dass er es nicht immer leicht mit mir hatte! Besonders hinweisen muss ich auch auf die unzähligen Strategien, welche er immer wieder aus dem Ärmel zu schütteln vermochte, um mir aus der Klemme zu helfen — diese Arbeit hätte ohne seine Lösungsvorschläge von mir wohl nie beendet werden können. Vielen Dank für Alles!

Bei der Gelegenheit möchte ich auch Peter M. Topping und Stefan Müller erwähnen. Dass Ideen von ihnen in meine Arbeit eingeflossen sind, ist ersichtlich. Aus diesem Grund bin ich umso stolzer, dass beide es akzeptiert haben, meine Dissertation zu begutachten.

Des weiteren seien Urs Lang, sowie Demetrios Christodoulou genannt, denn sie beide haben vor Jahren meine Freude an der Differentialgeometrie geweckt, sowie anschliessend genährt. Durch sie hatte ich es mir überhaupt in den Kopf gesetzt, meine Nase in die Mathematik zu stecken. Natürlich will ich dabei aber auch nicht Tom Ilmanen und Jürg Fröhlich vergessen, die mich — vermutlich ohne es zu wissen — immer wieder inspiriert haben.

Dem Schweizerischen Nationalfond möchte ich für die teilweise Finanzierung meiner Arbeit danken (Projekte Nr. 112009 & 121894), sowie dem Wolfgang Pauli Institut in Wien, insbesondere seinem Direktor Norbert J. Mauser, für das Finanzieren eines sechsmonatigen Aufenthalts in meiner Heimatstadt (im Rahmen des *thematic programme on "Control of (Nonlinear) Schrödinger Equations (2008)"*).

Dietmar Salamon sowie Michele Marcionelli möchte ich dafür danken, dass sie mir halfen, meine Bindung zur ETH aufrechterhalten zu können. Auch Christina Buchmann gebührt Dank in diesem Zusammenhang, sowie allgemein für ihre Ermutigungen.

Viele Kollegen haben mir mathematisch, aber auch persönlich, in den langen Jahren der Entstehung dieser Arbeit beigestanden. Von ETH und Uni seien hier unter anderen Jonas Hirsch, Thomas Huber, Alexandru Oancea, Johannes Sauter, Emanuele Spadaro, László Székelyhidi und Dominik Tasnady erwähnt.

Die Liste jener Leute, die mich moralisch unterstützt haben, ist lang. Besonders hervorgehoben seien: Gilles Angelsberg, Johanna Berghaus, Pasquale Cariglia, Valentina Georgoulas, Rik Harbers & Pamela Nitsch, Waltraud Höltl, Lukas Kaelin, Jeremy Salvadori, Christoph Schükro & Patricia Pühlhorn, sowie Josef Weinhäuser. Vielen Dank für Eure Freundschaft!

Die folgenden Personen standen mir unentwegt zur Seite: Theo Bühler, Georg Höltl, Ivo & Martina Kaelin–Bürgin, sowie Joachim Näf. Ohne Euch wäre gar nichts gegangen!

Bei Alina, Paula und Dorel Ostafe möchte ich mich ganz herzlich dafür bedanken, dass sie mich so schnell und unkompliziert in ihre wundervolle Familie aufgenommen haben.

Auch stehe ich bei meinem Vater, Peter René, sowie meiner Schwester, Estelle, in der Schuld: Keinen Augenblick haben sie je an mir gezweifelt, und immer haben sie mich unterstützt.

Zu guter Letzt möchte ich meiner Verlobten Lavinia danken, für ihren unermüdlichen Zuspruch, ihre immerwährende Hilfsbereitschaft und für ihre unerschöpfliche Liebe! Das Glück, Dich gefunden zu haben, ist unfassbar!